Wave-structure impact and rebound at the capillary scale and Faraday pilot waves

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Yves Couder

Faraday wave instability

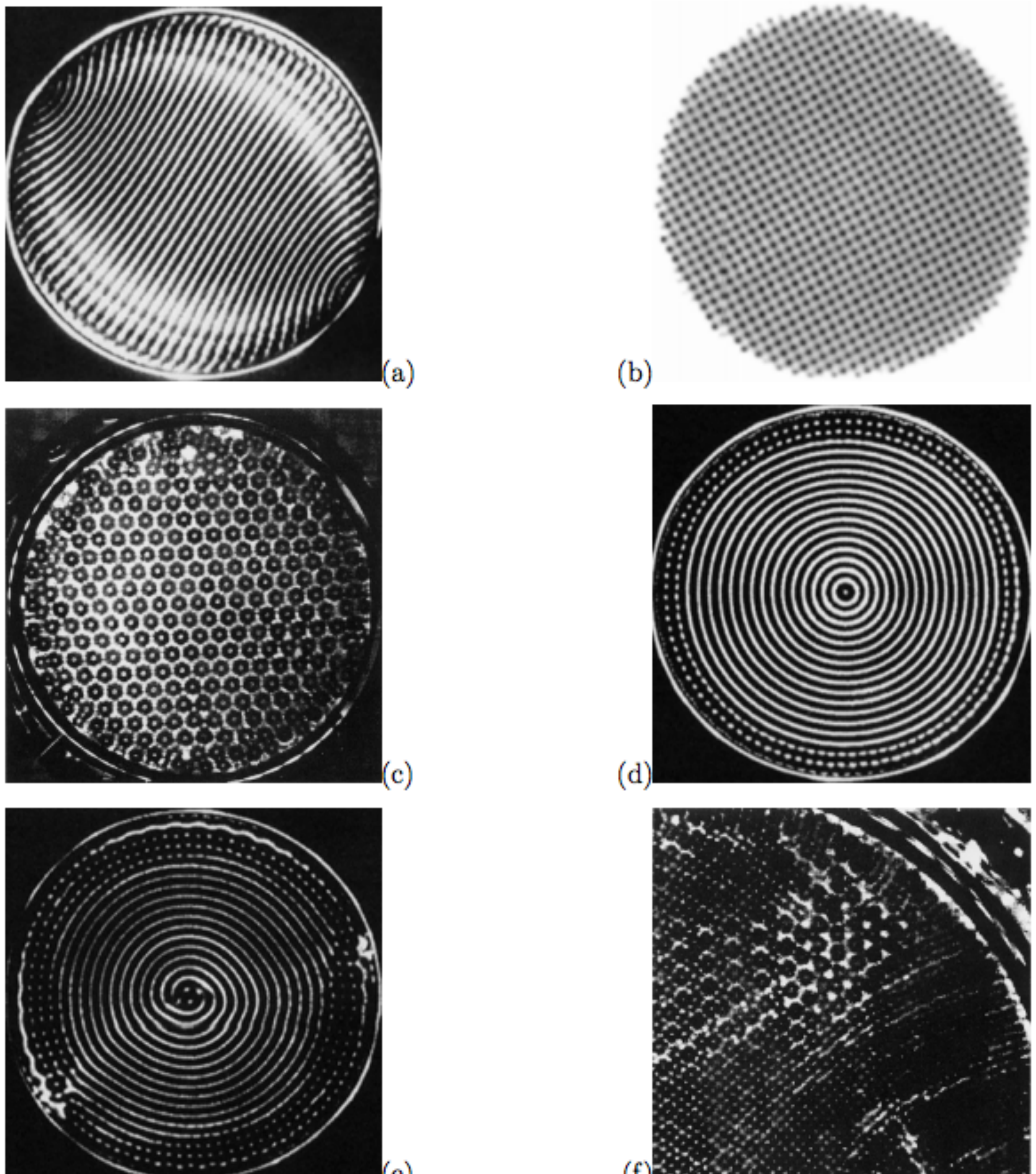
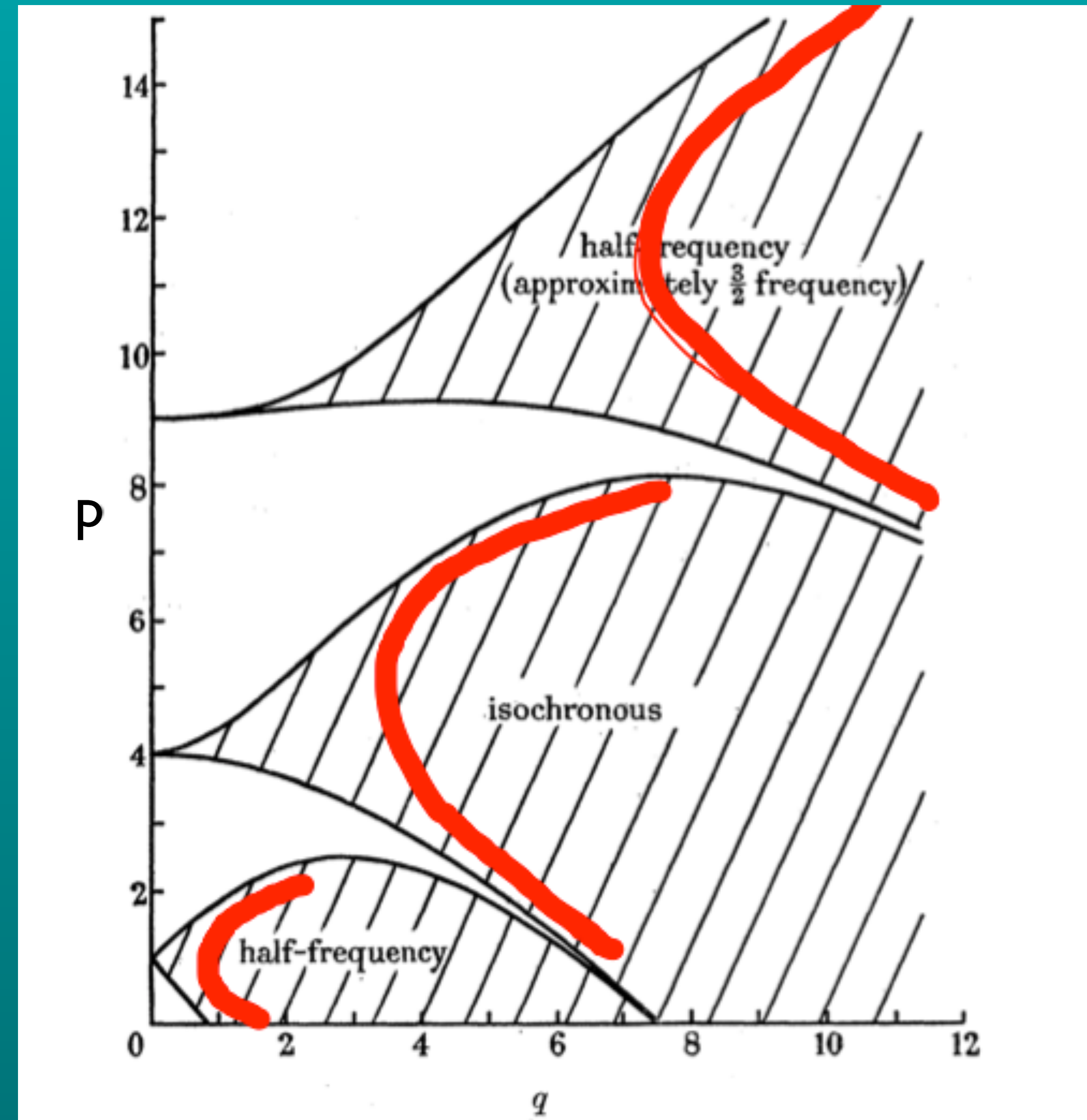


Figure 3.1: Faraday wave patterns observed in experiments with sinusoidal forcing. (a) Stripe pattern from [9]. (b) Square pattern from [63]. (c) Hexagonal pattern from [48]. (d) Target pattern from [9]. (e) Spiral pattern from [9]. (f) Region of coexisting squares and hexagons from [48].

Edwards & Fauve 1994

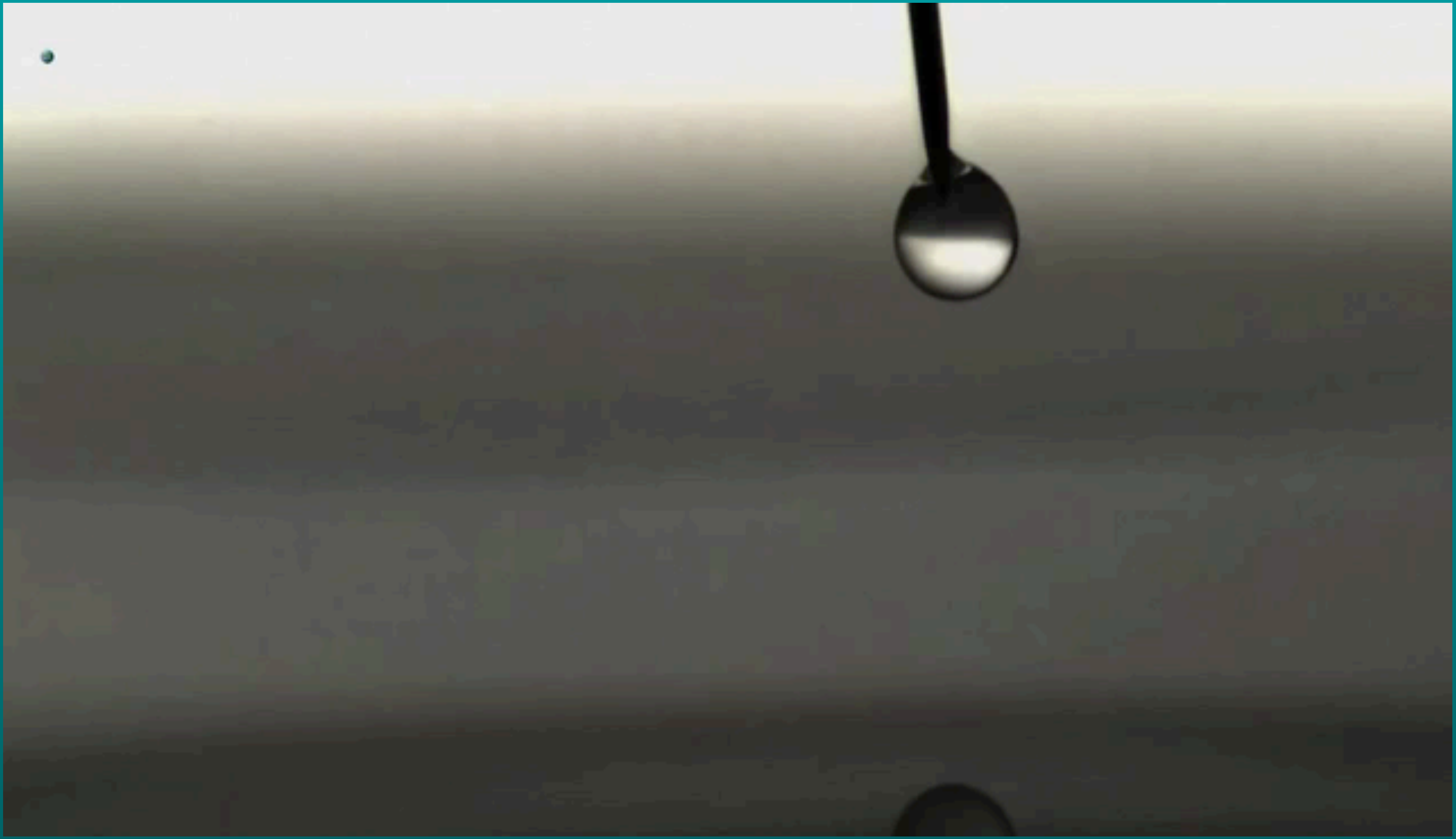


Benjamin & Ursell 1954

q = forcing strength

$$p = \left(\frac{\omega}{\omega_0/2} \right)^2$$

What happens when a drop falls into a bath?



Videos by John Bush, MIT.

Bouncing droplet on a Faraday stable vibrating bath

Videos by John Bush, MIT.

Walking droplet on vibrating bath: the Faraday Pilot Wave

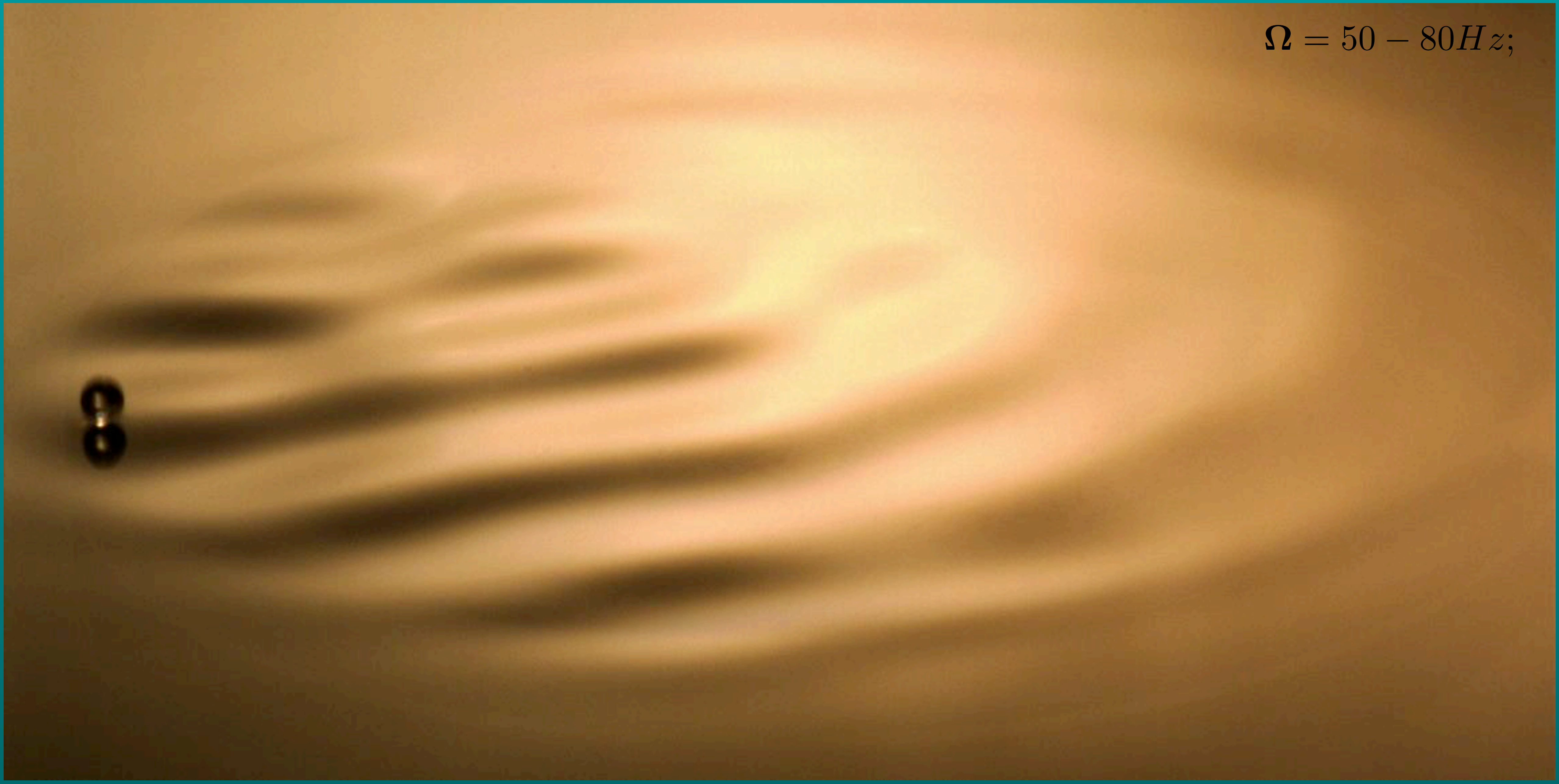
Videos by John Bush, MIT.

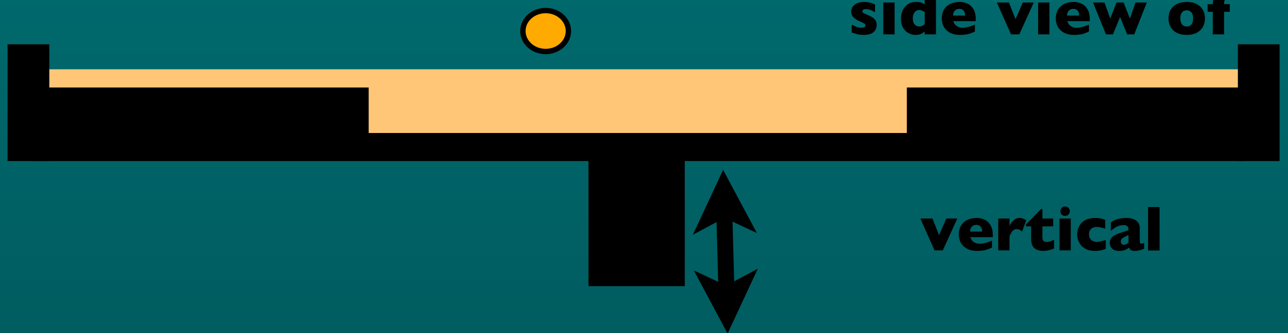
Strobed droplet propelled by its Faraday pilot wave

Videos by John Bush, MIT.

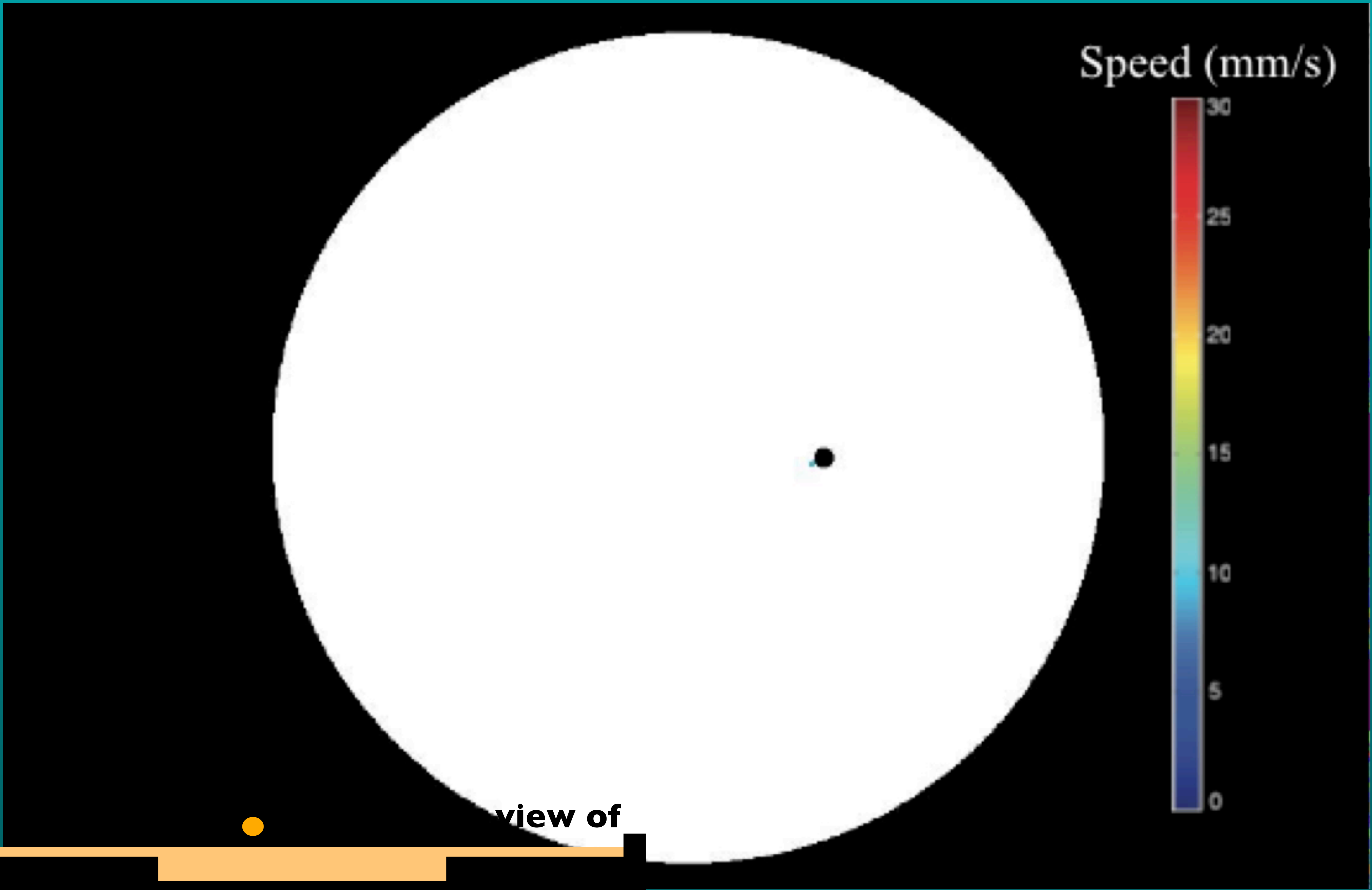
Confinement: particle in a corral

Videos by John Bush, MIT.



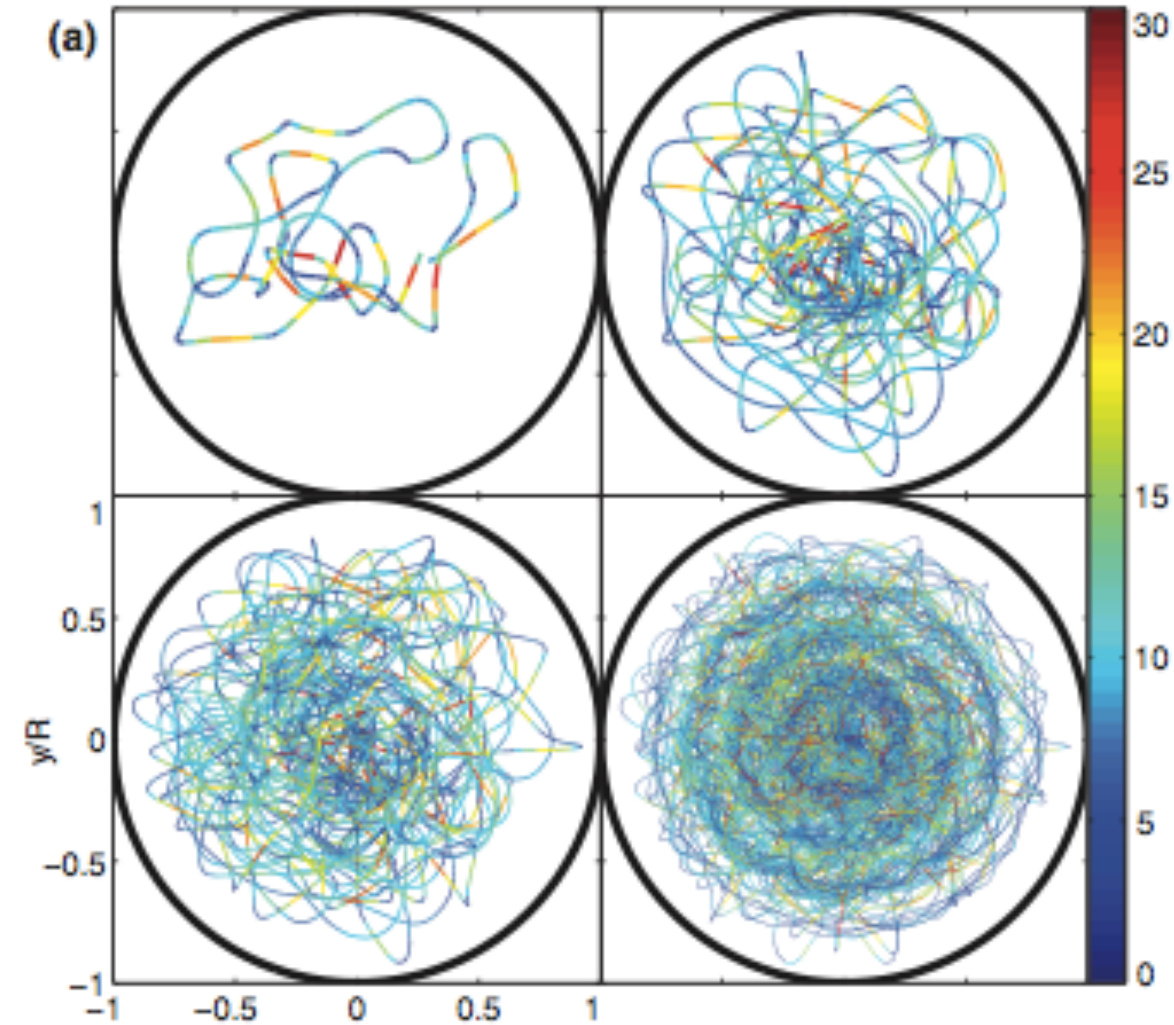


HQA: Corral Experiments - particle in a box

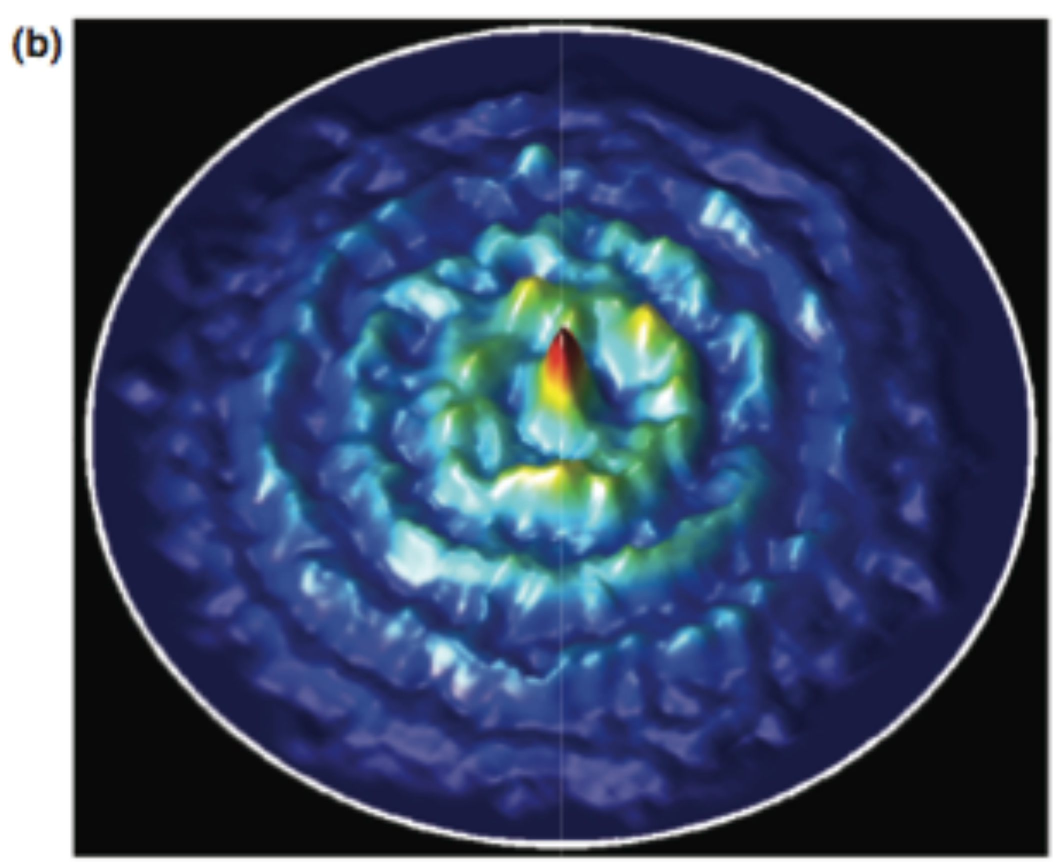


Harris et al Phys. Rev B 2013

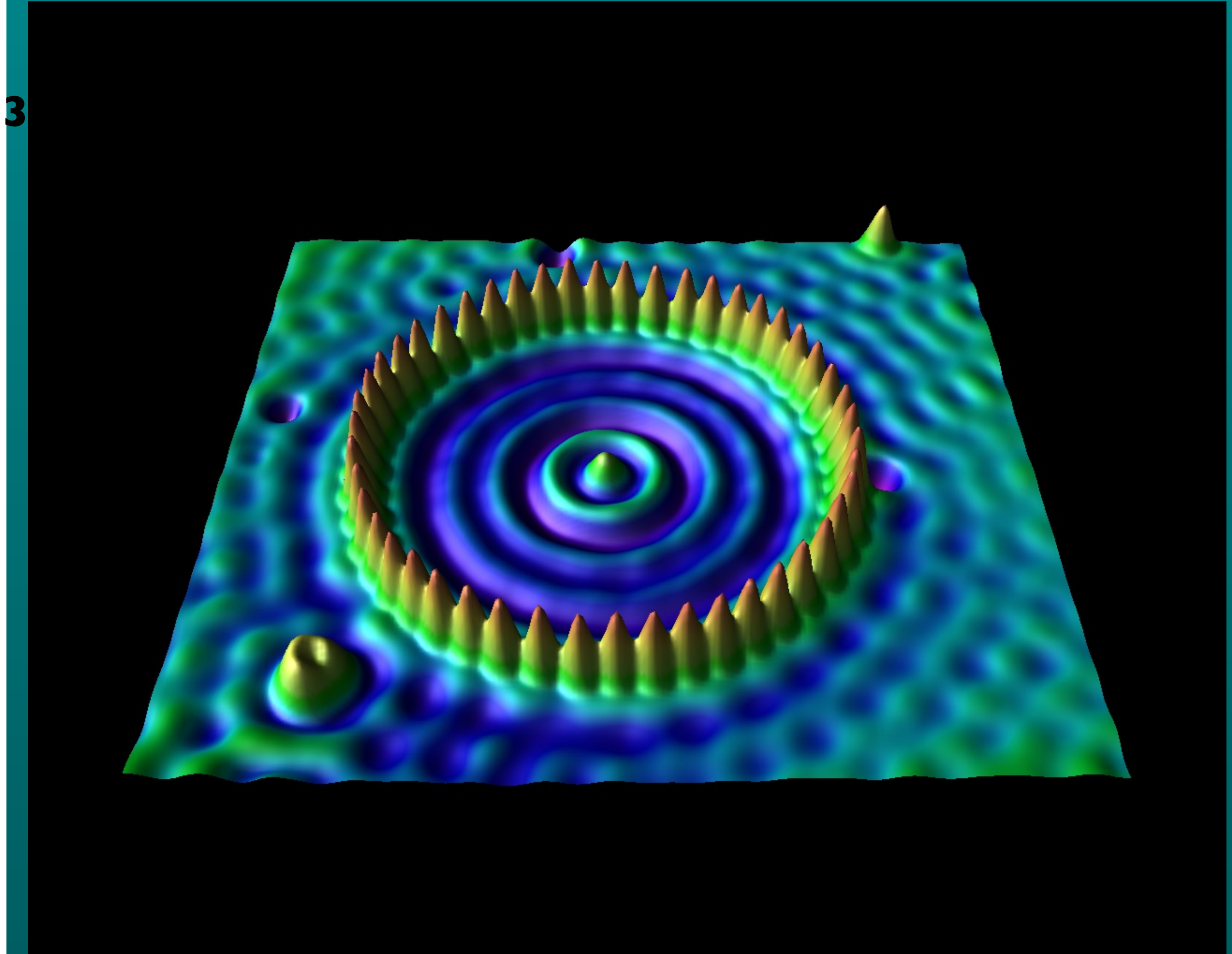
Droplet



Harris et al Phys. Rev B 2013



Electron



HQA: Particle in a potential

nature COMMUNICATIONS

ARTICLE

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Self-organization into quantized eigenstates of a classical wave-driven particle

Stéphane Perrard[1], Matthieu Labousse[2], Marc Miskin[1,2,†], Emmanuel Fort[2] & Yves Couder[1]

A growing number of dynamical situations involve the coupling of particles or singularities with physical waves. In principle these situations are very far from the wave particle duality at quantum scale where the wave is probabilistic by nature. Yet some dual characteristics were observed in a system where a macroscopic droplet is guided by a pilot wave it generates. Here we investigate the behaviour of these entities when confined in a two-dimensional harmonic potential well. A discrete set of stable orbits is observed, in the shape of successive generalized Cassinian-like curves (circles, ovals, lemniscates, trefoils and so on). Along these specific trajectories, the droplet motion is characterized by a double quantization of the orbit spatial extent and of the angular momentum. We show that these trajectories are intertwined with the dynamical build-up of central wave-field modes. These dual self-organized modes form a basis of eigenstates on which more complex motions are naturally decomposed.

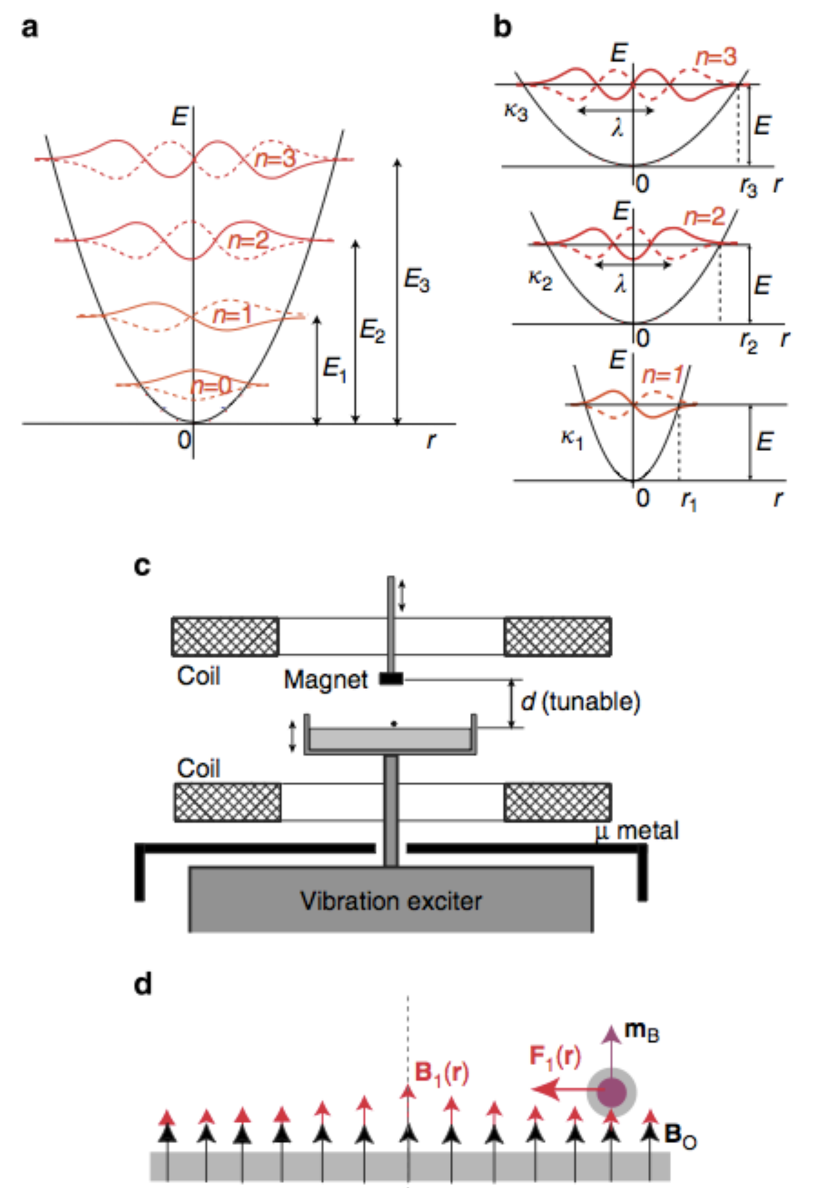
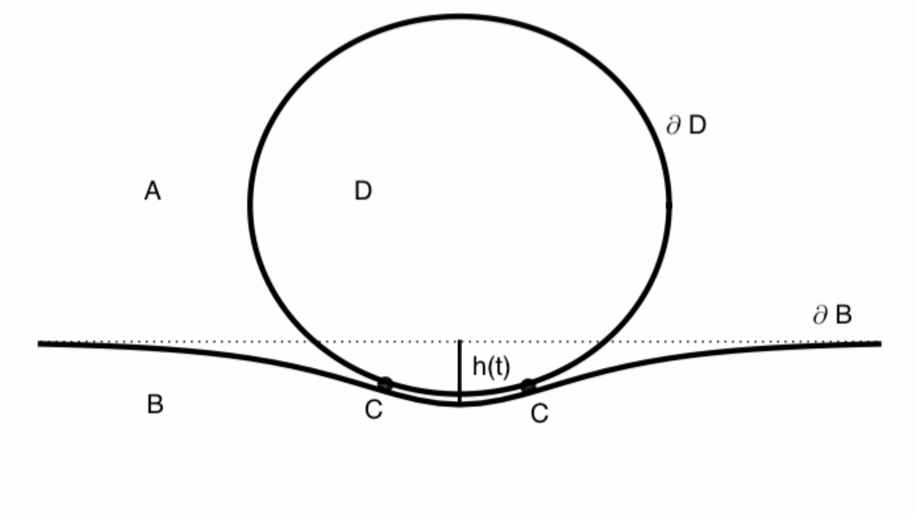


Figure 1 | Principle of the experiment and actual set-up. (**a**) Sketch of the successive eigenstates of increasing energy and decreasing wavelengths of a quantum particle in a one-dimensional harmonic potential well. (**b**) These

The Fluid Mechanics Problem

The full problem



The incompressible Navier Stokes

$$\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{u} + \mathbf{F}(t), \qquad \nabla\cdot\mathbf{u} = 0,$$

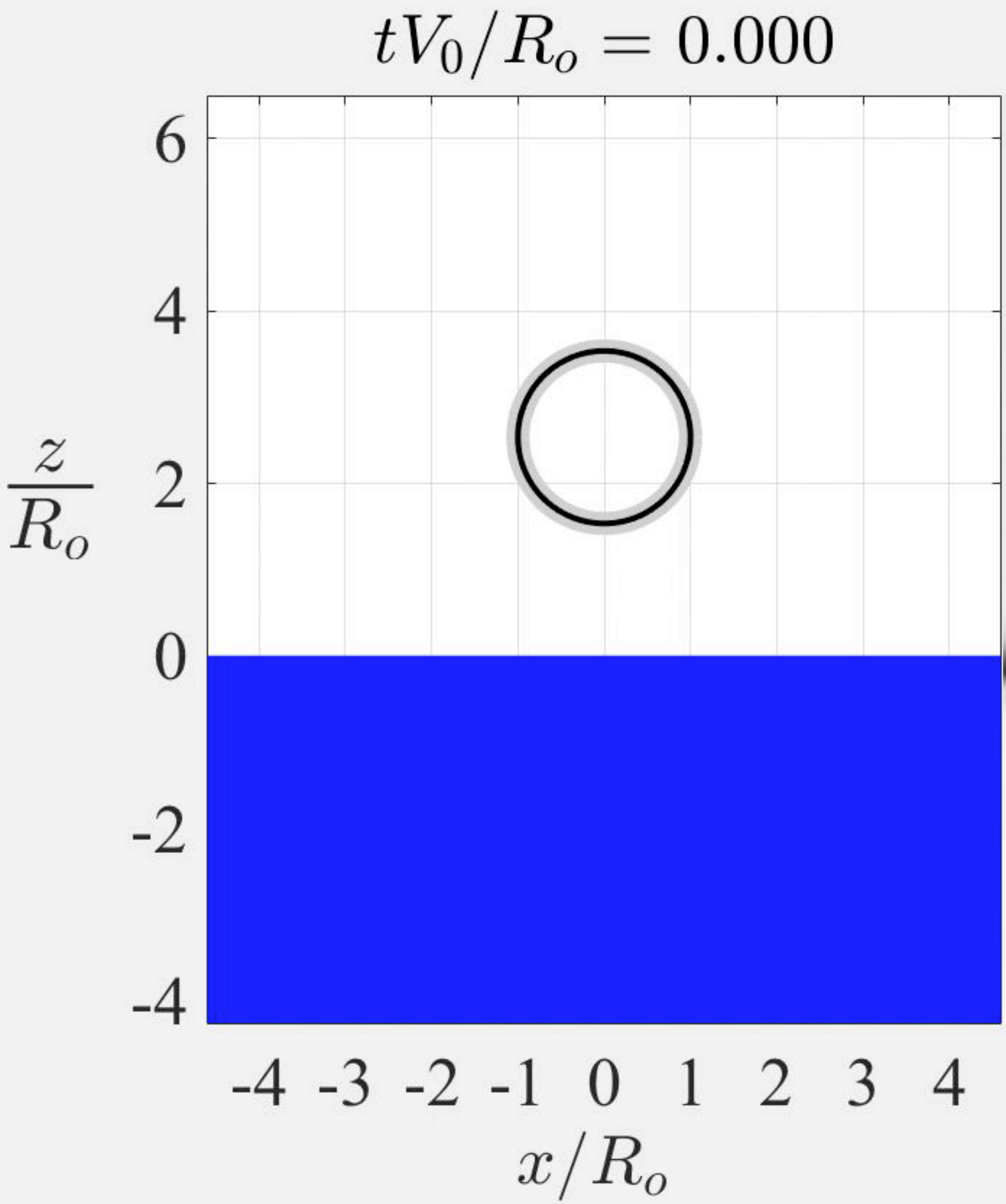
$$\mathbf{F} = -g(t)\mathbf{e_z} = -g(1-\Gamma\cos(\omega_0 t))\mathbf{e_z}$$

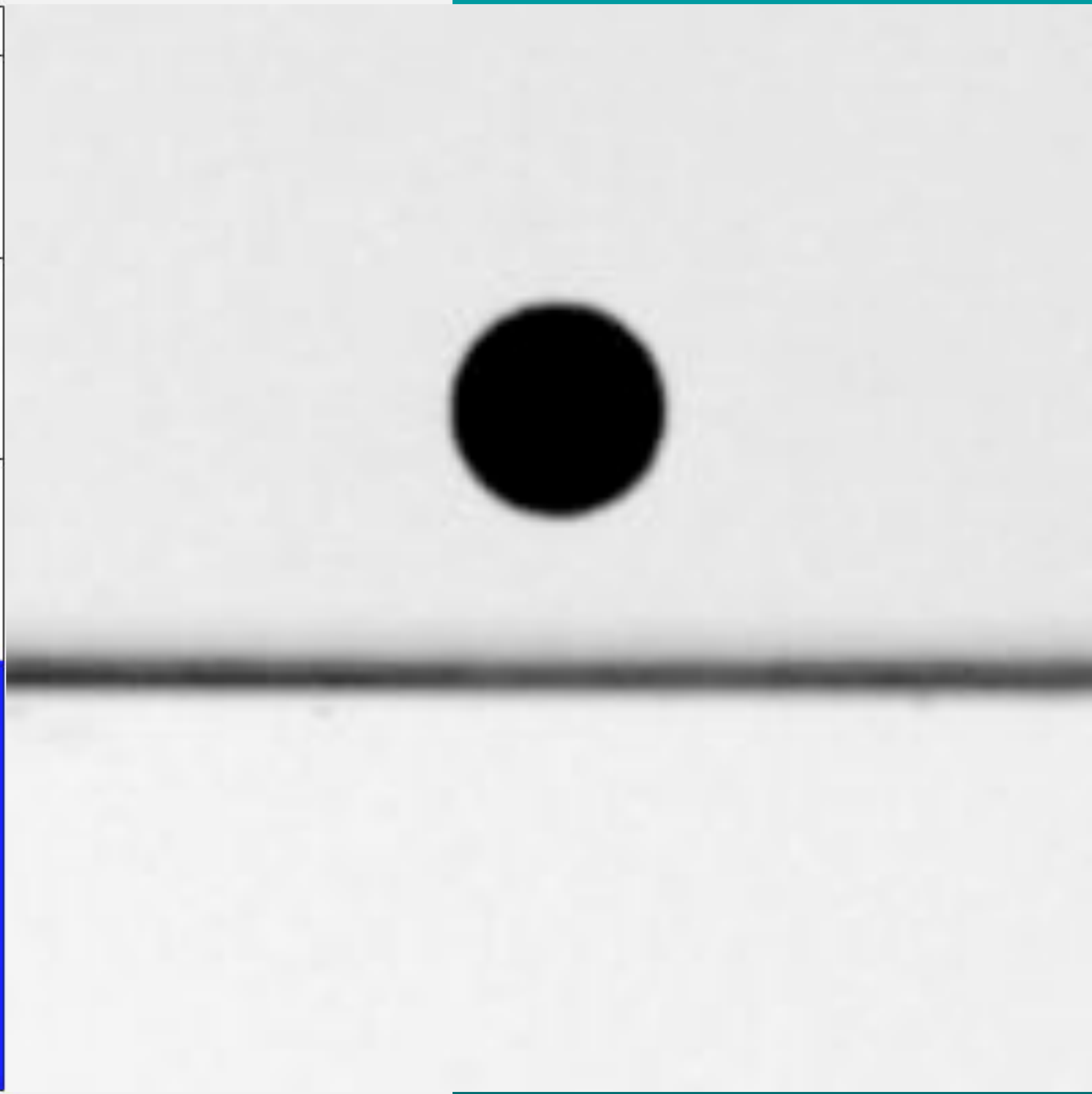
- ∂B and ∂D are defined by the equations $b(\mathbf{x},t) = 0$ and $d(\mathbf{x},t) = 0$

$$[p\,\mathbf{n} - \rho\nu\,\tau\cdot\mathbf{n}] = \sigma\kappa\mathbf{n}, \qquad [\mathbf{u}] = 0, \qquad D_t b = D_t d = 0.$$

- $[\cdot]$ denotes the jump of the quantity across the interface,
- D_t is the material derivative $\partial_t + \mathbf{u}\cdot\nabla$,
- σ the surface tension coefficient, κ is the mean curvature of the surface
- ν is the viscosity, ρ is the density (with different values in each domain).
- $\mathbf{n}$ is the unit normal and τ is the strain tensor $(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$.

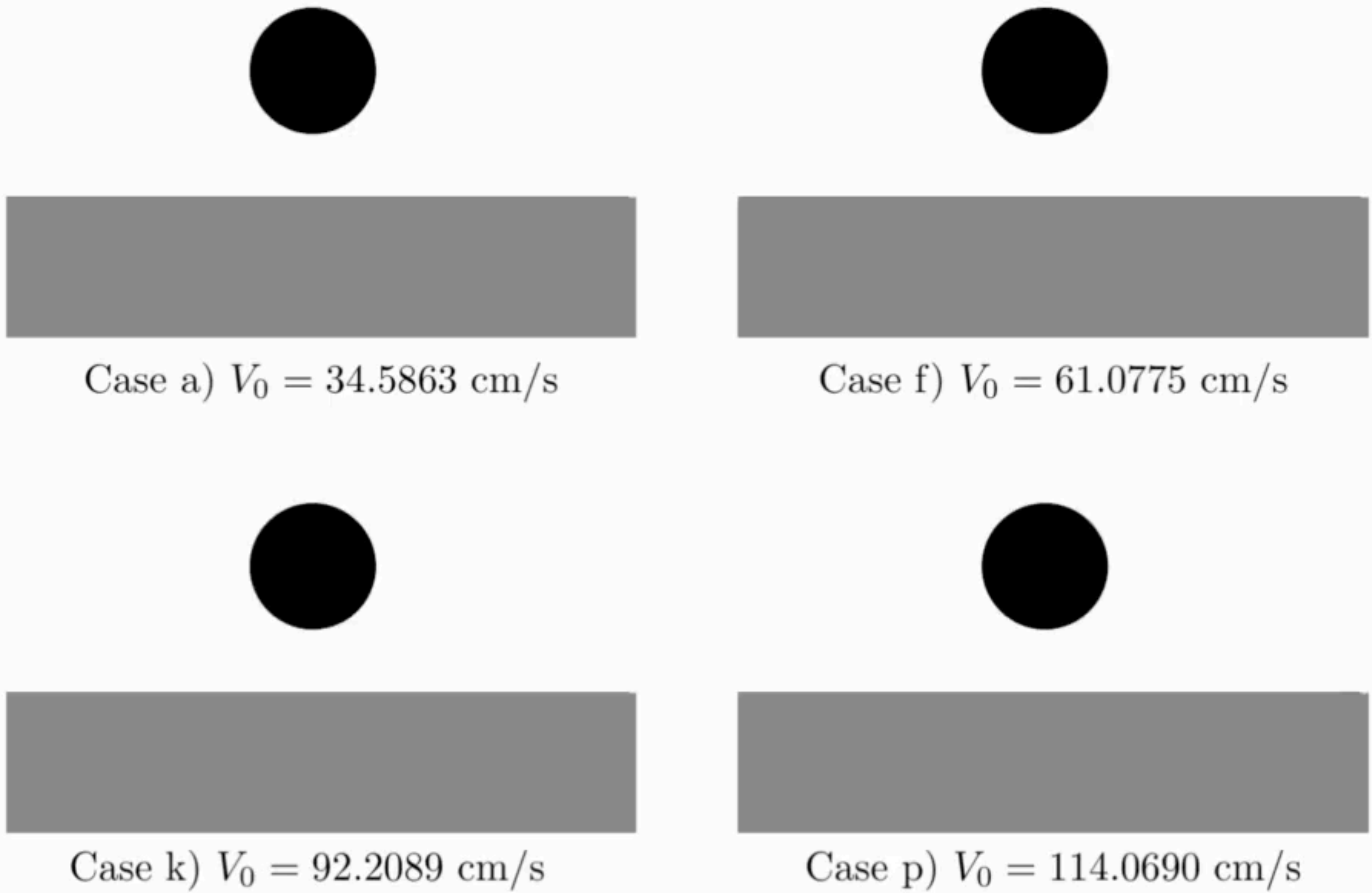
Single bounce of solid hydrophobic sphere





Video: Dan Harris

CFD (GERRIS) - axisymmetric



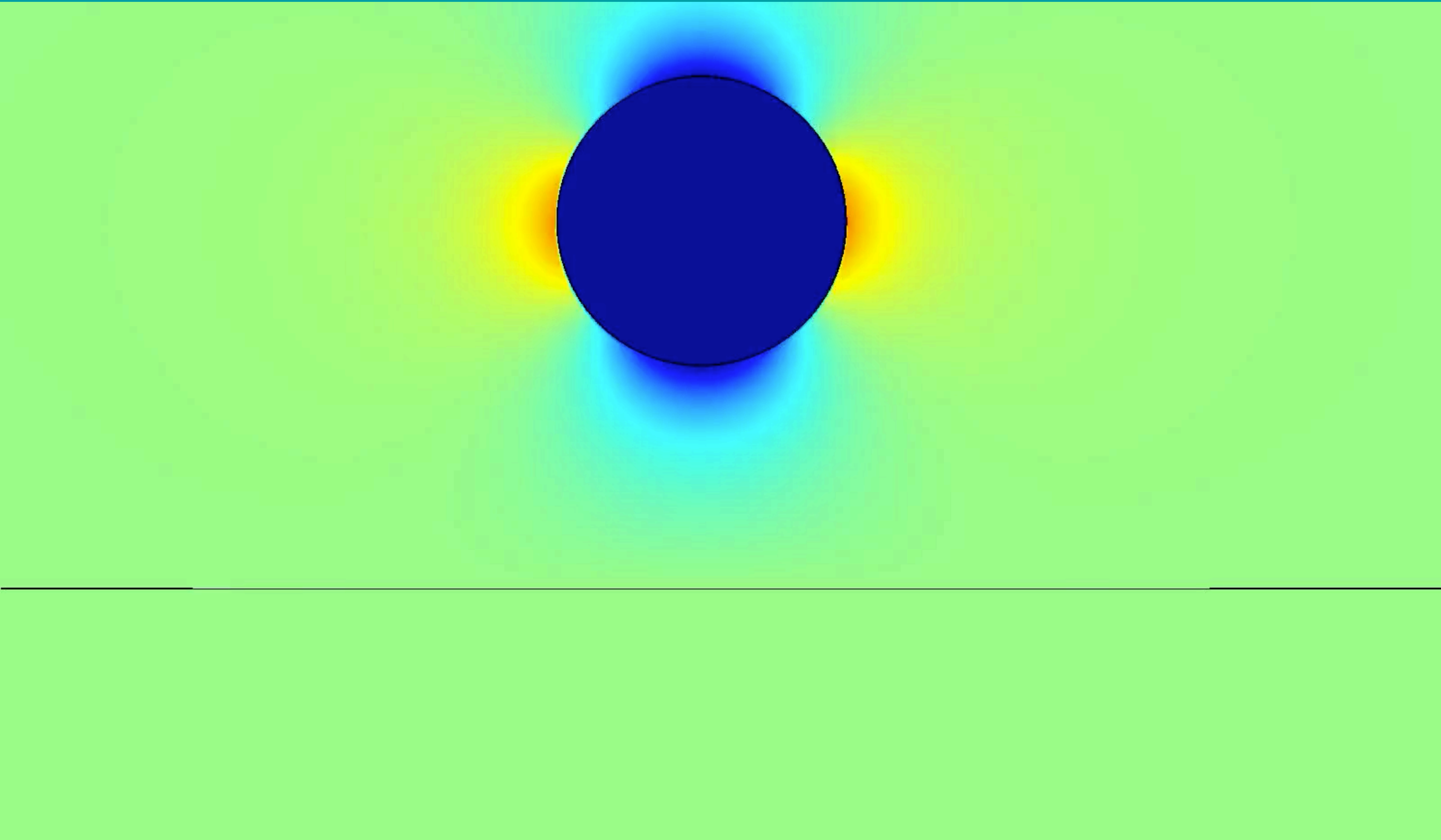
Case a) $V_0 = 34.5863$ cm/s

Case f) $V_0 = 61.0775$ cm/s

Case k) $V_0 = 92.2089$ cm/s

Case p) $V_0 = 114.0690$ cm/s

CFD (GERRIS) - axisymmetric



Joint work with Radu Cimpeanu.

Challenges

- Multiscale problem: length scales $\sim 10^{-6}m.$ to $\sim 10^{-1}m.$; timescales $\sim 10^{-2}s.$ to $\sim 10^{3}s.$
- Free boundary problem: bath surface and droplet deform.
- Modelling is necessary: both for feasibility and to extract the important effects.
- Removing the lubrication layer results in a "nonsmooth" (piecewise smooth) dynamical system.

Modelling Approaches

- Discrete Waves: Eddi, Couder, Bush & others: Wavefield as a sum of discrete single Bessel (or simpler) standing waves with time decay.
- Trajectory Equation: Rosales, Oza & Bush: Discrete sum approximated by an integral, makes analysis possible.
- Wave Generation: M. et. al.: Continuous-time bath-droplet interaction - droplet as a wavemaker - captures further important effects (e.g. Doppler, (m,n) modes, decay).
- Discrete Wave Generation: Durey & M.: Discrete-time bath-droplet interaction - allows for fast realistic simulations and analysis.

Discrete Impact Model
(Durey & Milewski 2017)

Simplifying assumptions

Assume:

- Period doubled vertical dynamics (i.e. (2,1) mode).
- Instantaneous impacts: $f(t) = f_0\,\delta(t - t_n)$.
- Impacts occur at a point: $P(\mathbf{x}, t) = f(t)\delta(\mathbf{x} - \mathbf{X}(t))$.
- The result: $f_0 = gT$, where $T = 4\pi/\omega_0$ is the time between impacts.

Fourier-Hankel transform introduces orthogonal basis functions:

$$\Phi_m(r, \theta; k) = J_m(kr)\cos(m\theta),$$
$$\Psi_m(r, \theta; k) = J_m(kr)\sin(m\theta),$$

for all $k \in \mathbb{R}^+$ and for all $m \in \mathbb{N}$.
Obtain system of homogeneous ODEs with jump conditions.

Dynamics and jump conditions

$$\eta(r,\theta,t) = \sum_{m=0}^{\infty} \int_0^{\infty} k\Big(a_m(t;k)\Phi_m(r,\theta;k) + b_m(t;k)\Psi_m(r,\theta;k)\Big)\, dk.$$

During flight:

$$\mathcal{L}_k a_m(t;k) = 0, \qquad \mathcal{L}_k b_m(t;k) = 0.$$

$\mathcal{L}_k$ is a damped Mathieu differential operator.

$$\mathbf{X}''(t) + V'\left[\mathbf{X}(t)\right] = \mathbf{0}.$$

At impacts:

$$[a_m'(t_n;k)]_-^+ = -P_m(k)\Phi_m\big(\mathbf{X}(t_n);k\big),$$
$$[b_m'(t_n;k)]_-^+ = -P_m(k)\Psi_m\big(\mathbf{X}(t_n);k\big).$$

$$[\mathbf{X}'(t_n)]_-^+ = -F(c)\left(\frac{1}{c}\sqrt{\frac{B}{R}}\nabla\eta(\mathbf{X}(t_n),t_n) + \mathbf{X}'(t_n)\right)$$

Regular walking states - orbits, walkers, pairs, trains

Walking states are found under a *periodicity under shift* condition.

Period Map = Mathieu Map ∘ Graf Addition Map ∘ Jump Conditions Map

- Find discrete-time travelling solutions with speed δx.
- Linearise map to analyse stability of steady states.
- Lower wave field energy for walking than unstable bouncing.
- Wave field has exponential spatial decay and Doppler shift.

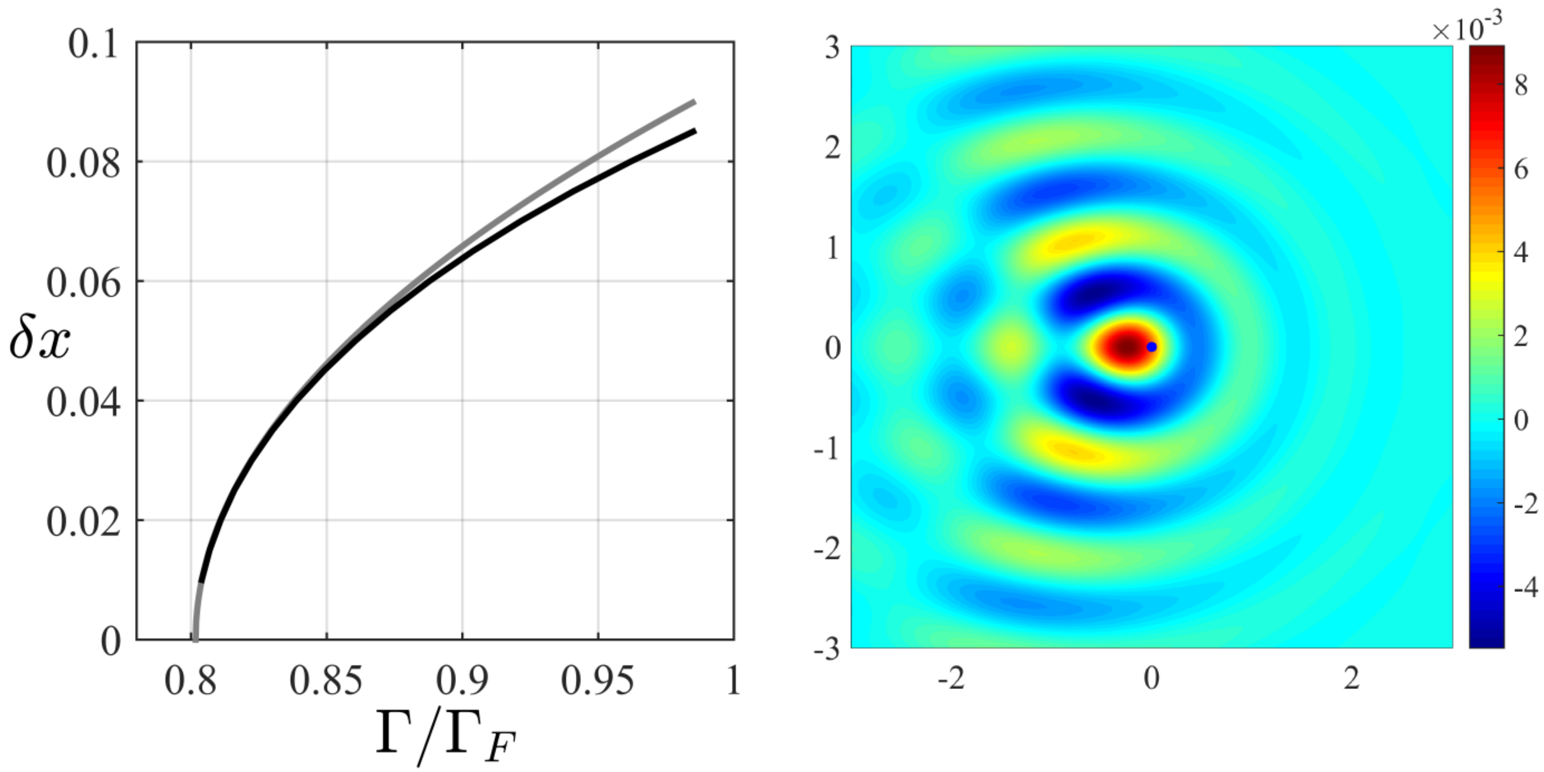
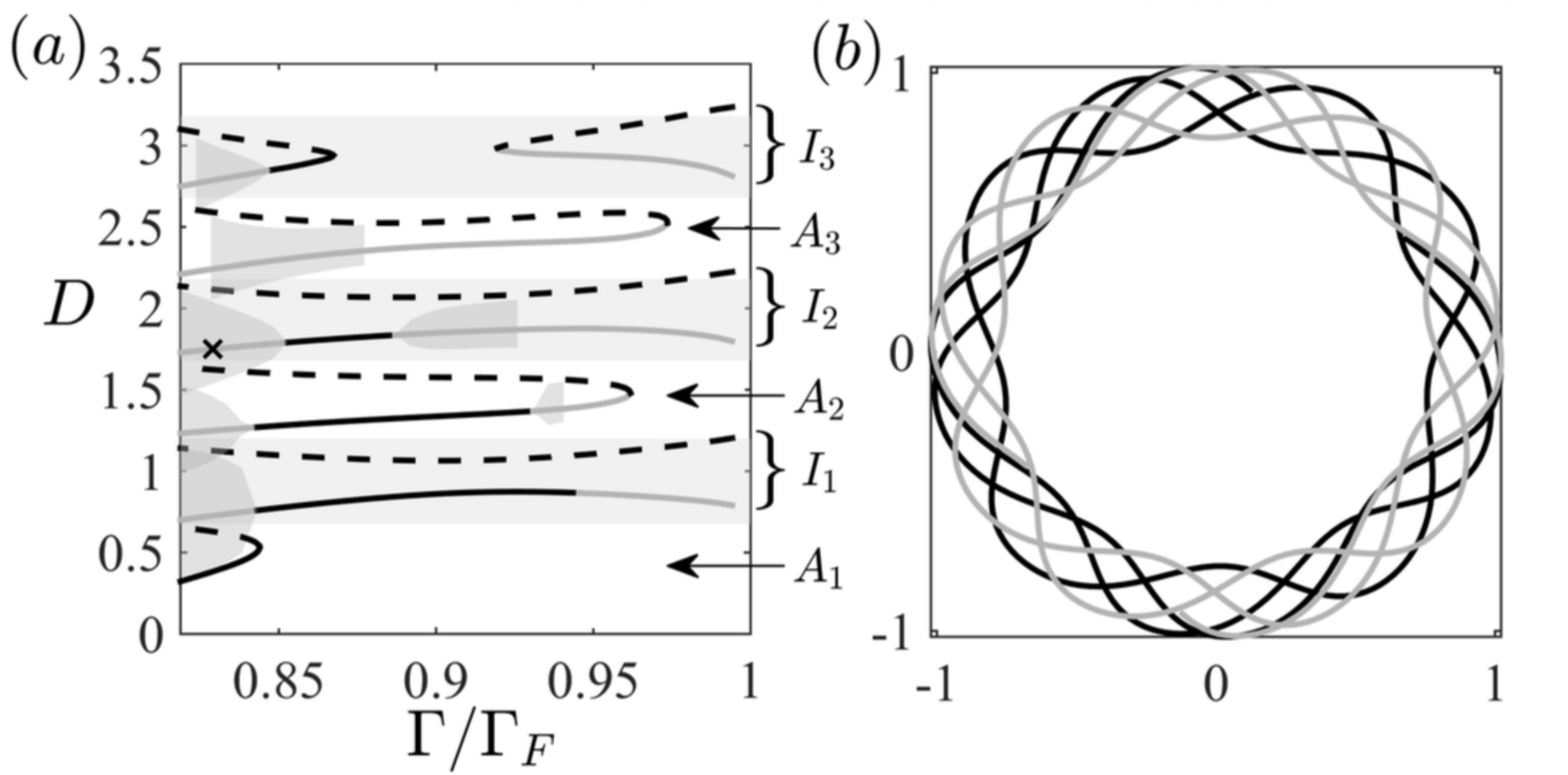


Figure: Left: full solution (black) with analytical approximation (grey) valid for $\delta x \ll 1$. Right: Wave field for $\delta x = 0.08$ and $\Gamma/\Gamma_F = 0.96$.

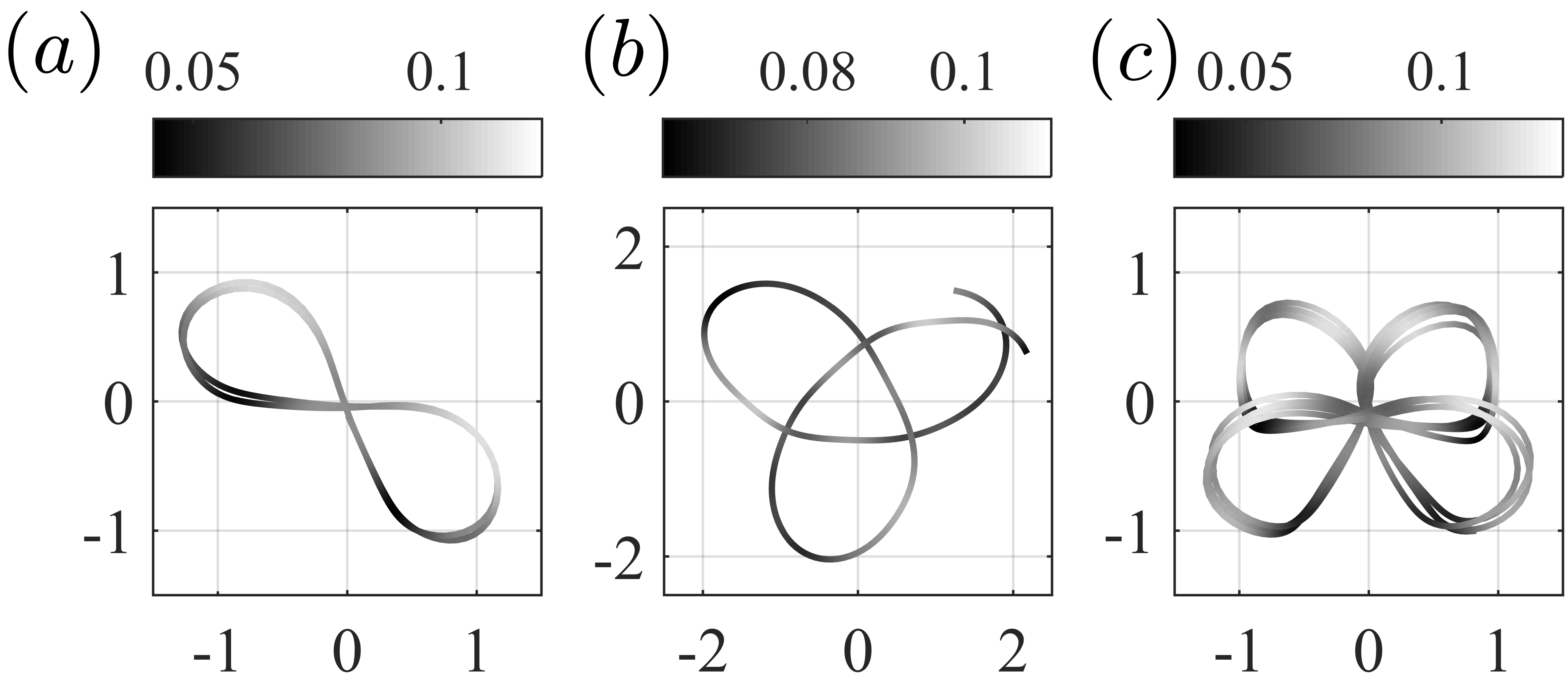
Two orbiting droplets - quantization

- In-phase and out of phase orbiters obtained exactly. Map composition also gives stability.
- Obtain stable, weakly unstable (Hopf), strongly unstable solutions.



Exotic orbits

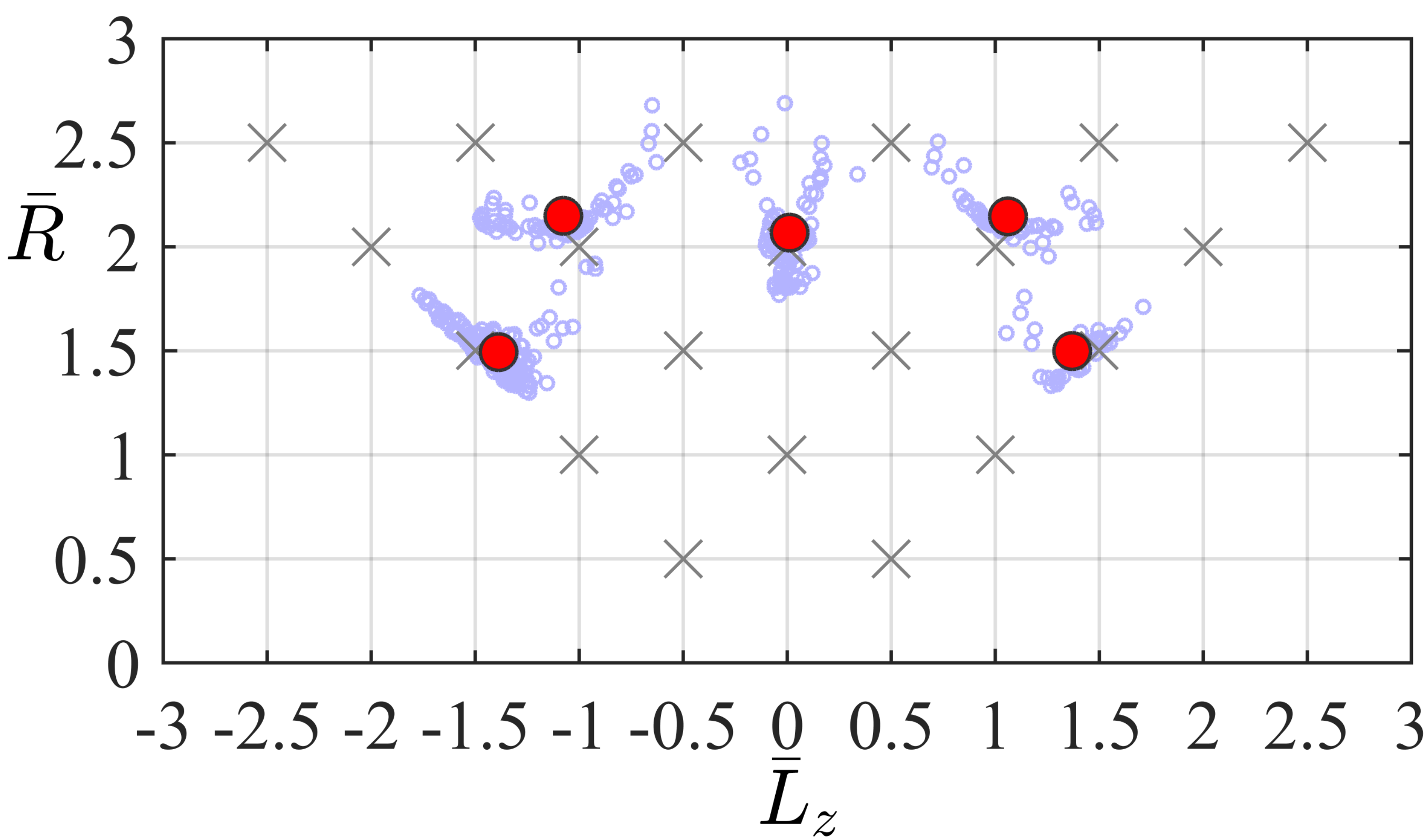
- Circular orbits destabilize to more exotic orbits (lemniscate, trefoil, butterfly).
- Experimentally observed double quantization in $\bar{R}$ and $\bar{L}_z$ (Perrard et al 2014).

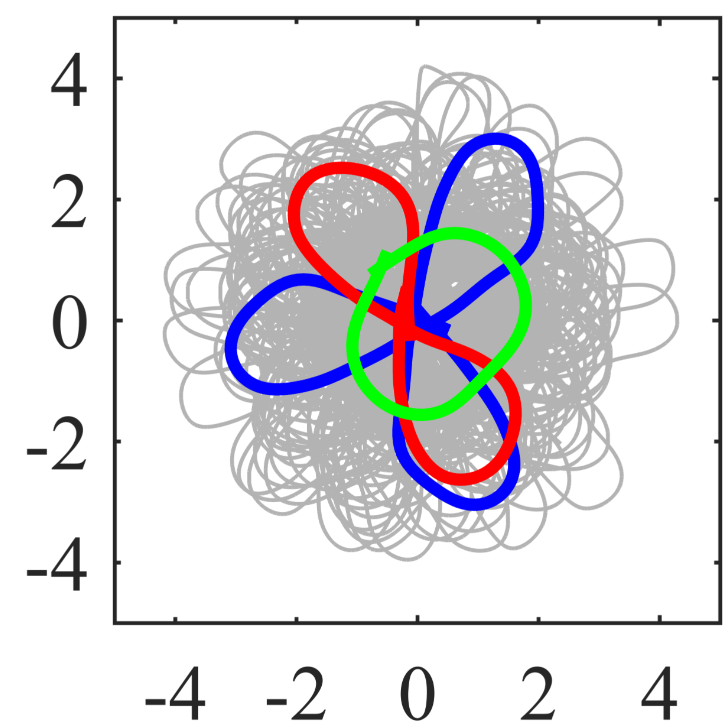


Statistical analysis of chaotic trajectories

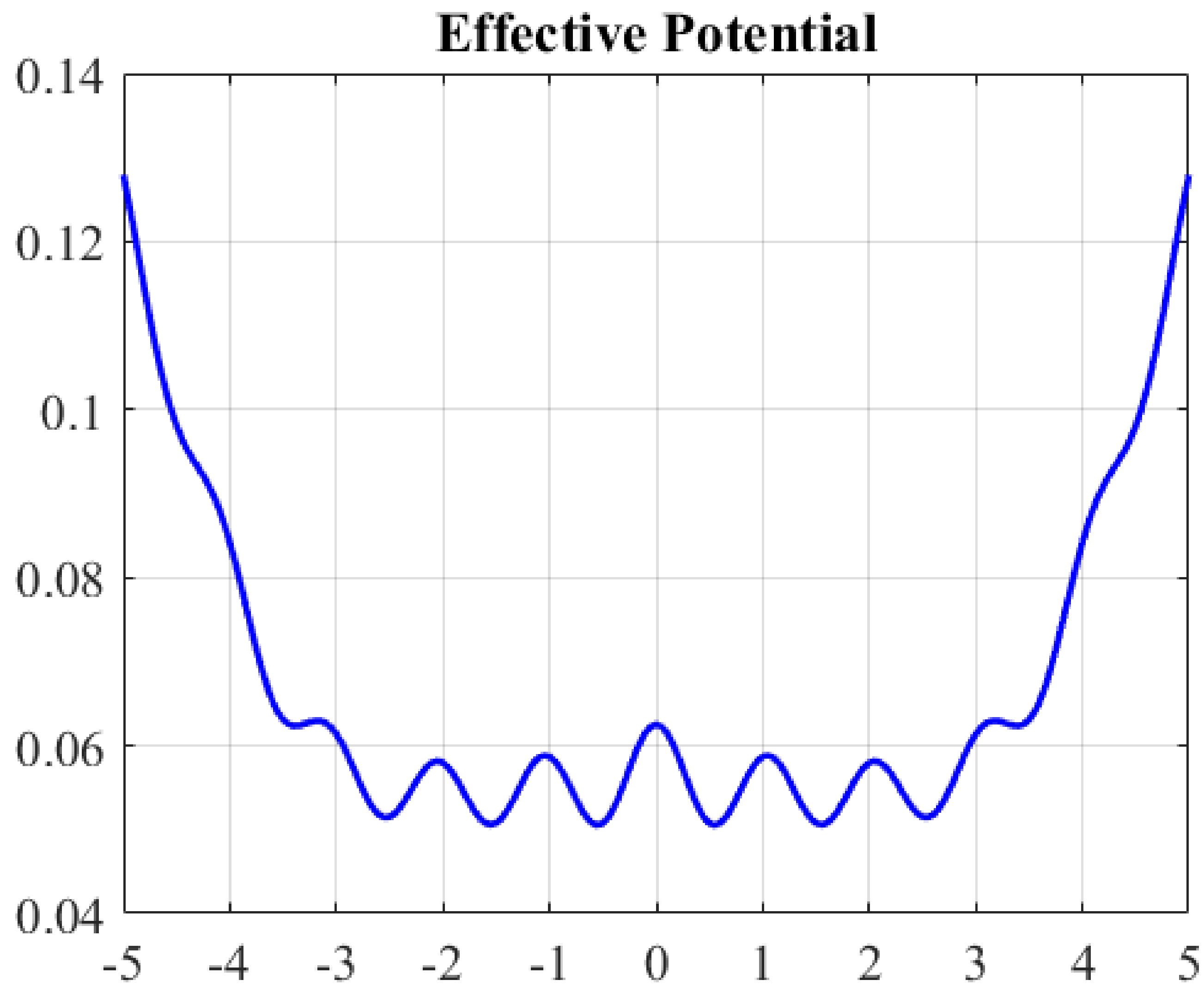
Simulate long trajectory starting from (unstable) circular orbit.
Segment trajectory at points of maximum radius and compute mean radius $\bar{R}$ and angular momentum $\bar{L}_z$ over each sub-trajectory.
Compute cluster centroids using K-means clustering.

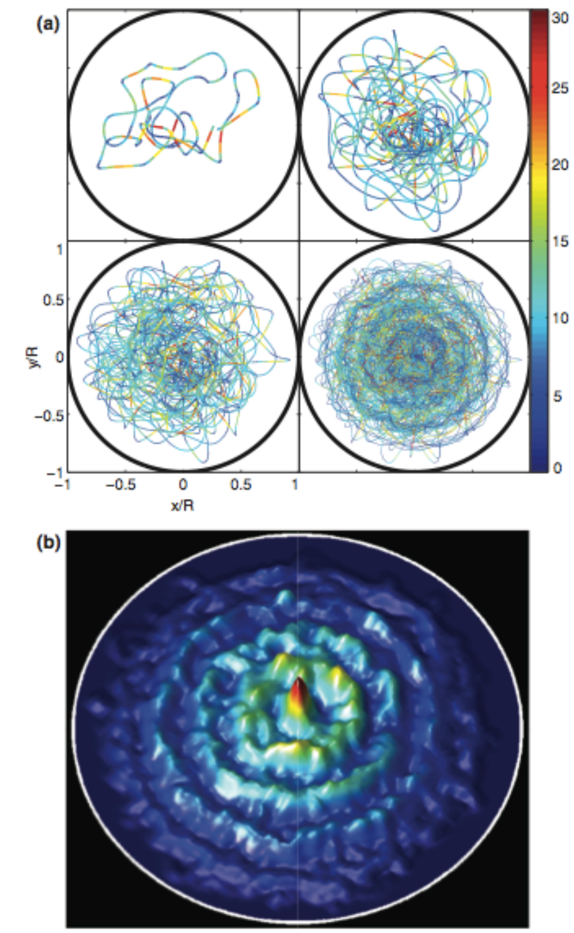
$\rightsquigarrow$**Double Quantization in $\bar{R}$ and $\bar{L}_z$!**





- Effective potential $\equiv$ mean wave field + harmonic potential.
- Gradient of effective potential corresponds to directions of mesh.





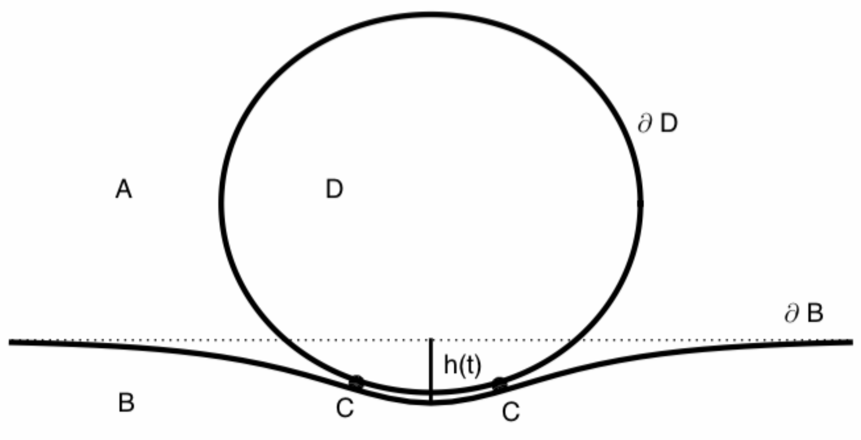
Theorem 1. *The stationary probability distribution $\mu(\boldsymbol{x})$ for the droplet position and the mean wave field at impact $\bar{\eta}(\boldsymbol{x})$ are related by*

$$\bar{\eta}(\boldsymbol{x}) = \int_{\mathbb{R}^2} \eta_B(\boldsymbol{x}-\boldsymbol{y})\mu(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y} = (\eta_B * \mu)(\boldsymbol{x}), \tag{6}$$

where $\eta_B(\boldsymbol{x})$ is the radially symmetric wave field of a bouncer centred at the origin.[1]

Continuous-time wave generation and nonlinear spring model for bouncing

Modelling the impact



Consider the droplet to be centred at $(\mathbf{X}(t), Z(t))$, and a free surface $\eta(x, t)$.

$$m\ddot{Z} = -mg(t) + \left\{-\alpha_f \dot{Z}\right\} I_f(t) + \mathcal{F}\left(Z, \dot{Z}, \eta, \eta_t\right) I_i(t),$$

$$m\ddot{\mathbf{X}} = \left\{-\alpha_f \dot{\mathbf{X}}\right\} I_f(t) + \left\{-\alpha_i \dot{\mathbf{X}} - \nabla\bar{\eta}|_{\mathbf{x}=\mathbf{X}}\right\} \mathcal{F}\; I_i(t),$$

- I_f and I_i are indicator functions of flight and impact
- $\mathcal{F}\left(Z, \dot{Z}, \bar{\eta}, \bar{\eta}_t\right)$ is the force normal to the free surface exerted on the droplet.
- m is the droplet mass, α_f and α_i are flight and impact drag coefficients
- Replacing η by $\bar{\eta}$ in $\mathcal{F}$, an average representation of the bath's free surface

(notion of a "penetration depth" $h(t) = |Z - R - \bar{\eta}|$)

Modelling - the waves

Linearised quasi-potential approximation for weakly damped waves

$$\begin{aligned}
\Delta\phi &= 0, \qquad z<0,\\
\phi_t &= -g(t)\,\eta + \frac{\sigma}{\rho}\Delta_H\eta + 2\nu\Delta_H\phi - \frac{1}{\rho}P_D(\mathbf{x}-\mathbf{X}(t),t), \qquad z=0,\\
\eta_t &= \phi_z + 2\nu\Delta_H\eta, \qquad z=0,\\
P_D &= \mathcal{F}(t)I(|\mathbf{x}-\mathbf{X}|<R(t))/\pi R(t)^2
\end{aligned}$$

- ϕ is the potential in a Helmholtz decomposition $\mathbf{u} = \nabla\phi + \nabla\times\mathbf{\Psi}(\phi)$.
- $R(t)$ is a contact radius - modelled geometrically $R(Z-\bar{\eta})$.
- Δ_H is the horizontal Laplacian.

Modelling the impact (cont)

- $I_i(t)$ must be calculated from geometric information on the waves and flight of drop.
- The dynamical system is now non-smooth: there is a switch between impact and flight.

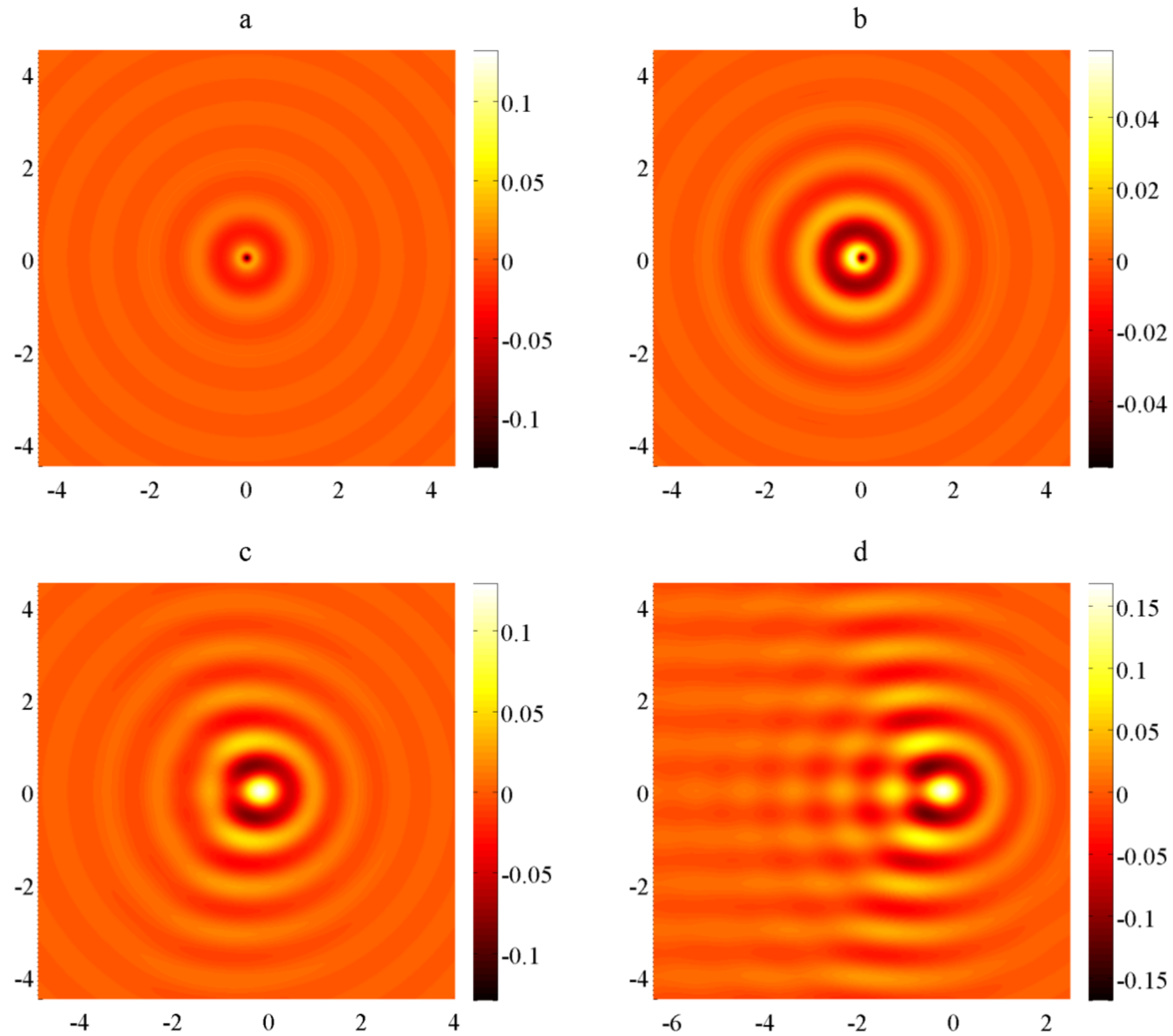
One possible vertical dynamics (ie a model for $\mathcal{F}$) (see Molacek and Bush 2014) is:

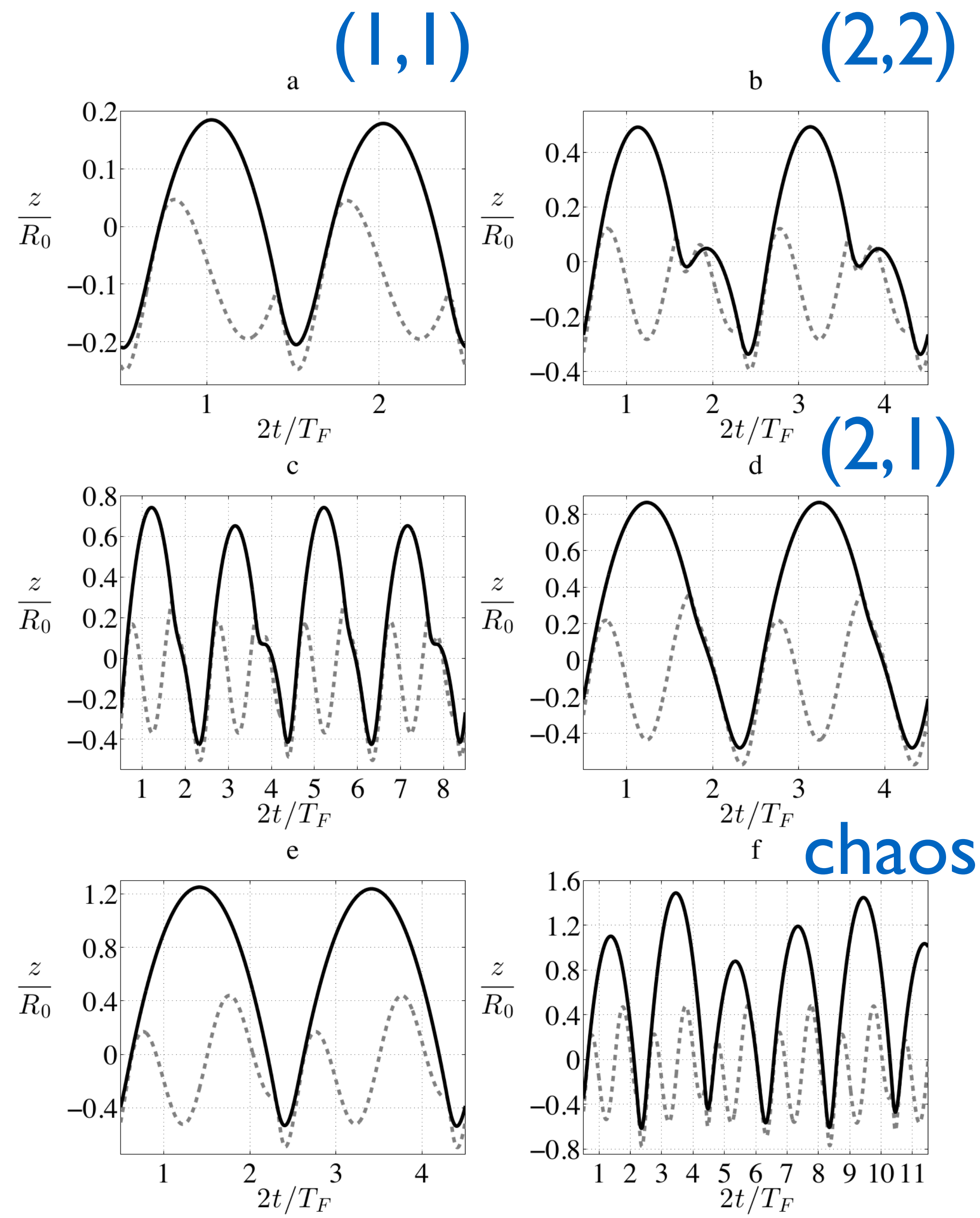
$$m\ddot{Z} = -mg(t),$$

$$\left(1 + \frac{c_3}{\ln^2\left|\frac{c_1 R_0}{Z-\bar{\eta}}\right|}\right) m\ddot{Z} + \frac{4}{3}\frac{\pi\mu R_0 c_2}{\ln\left|\frac{c_1 R_0}{Z-\bar{\eta}}\right|}\left(\dot{Z} - \dot{\bar{\eta}}\right) + \frac{2\pi\sigma}{\ln\left|\frac{c_1 R_0}{Z-\bar{\eta}}\right|}(Z - \bar{\eta}) = -mg(t),$$

during flight and impact respectively.

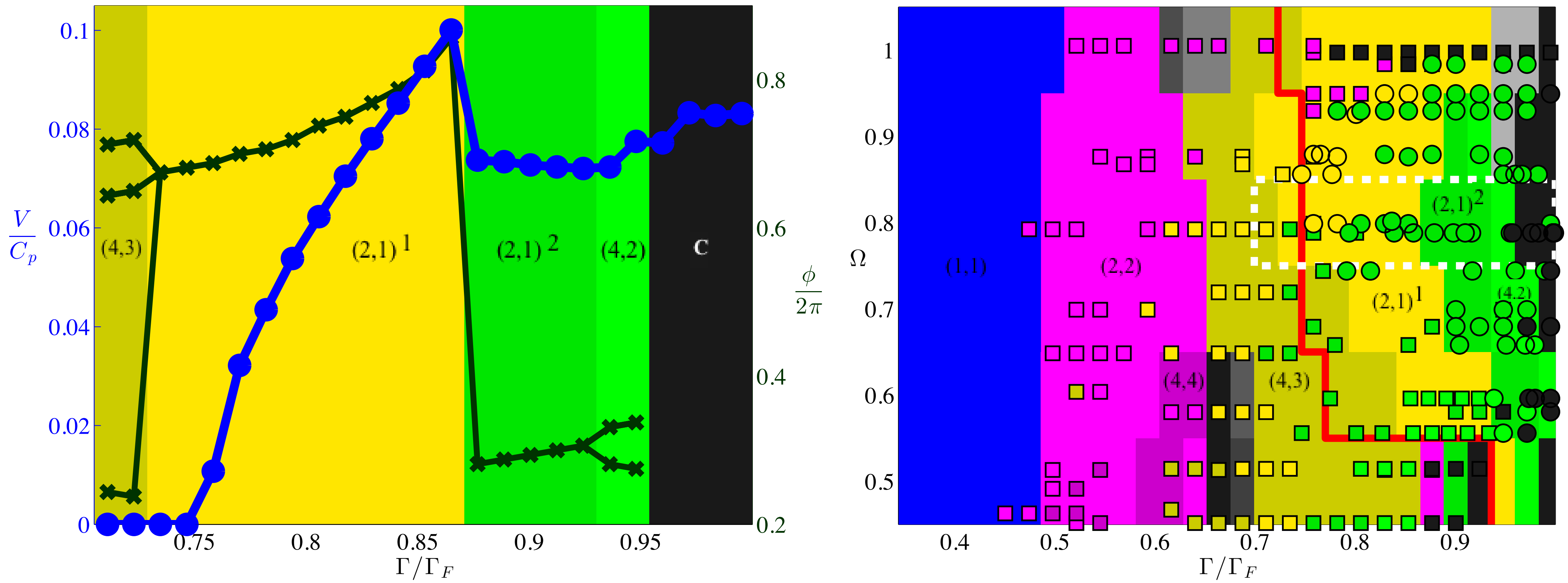
Complex bouncing and walking





$(m, n)^p$ states: m is number of forcing periods, n is number of bounces, p is an energy state. The control parameter is Γ.

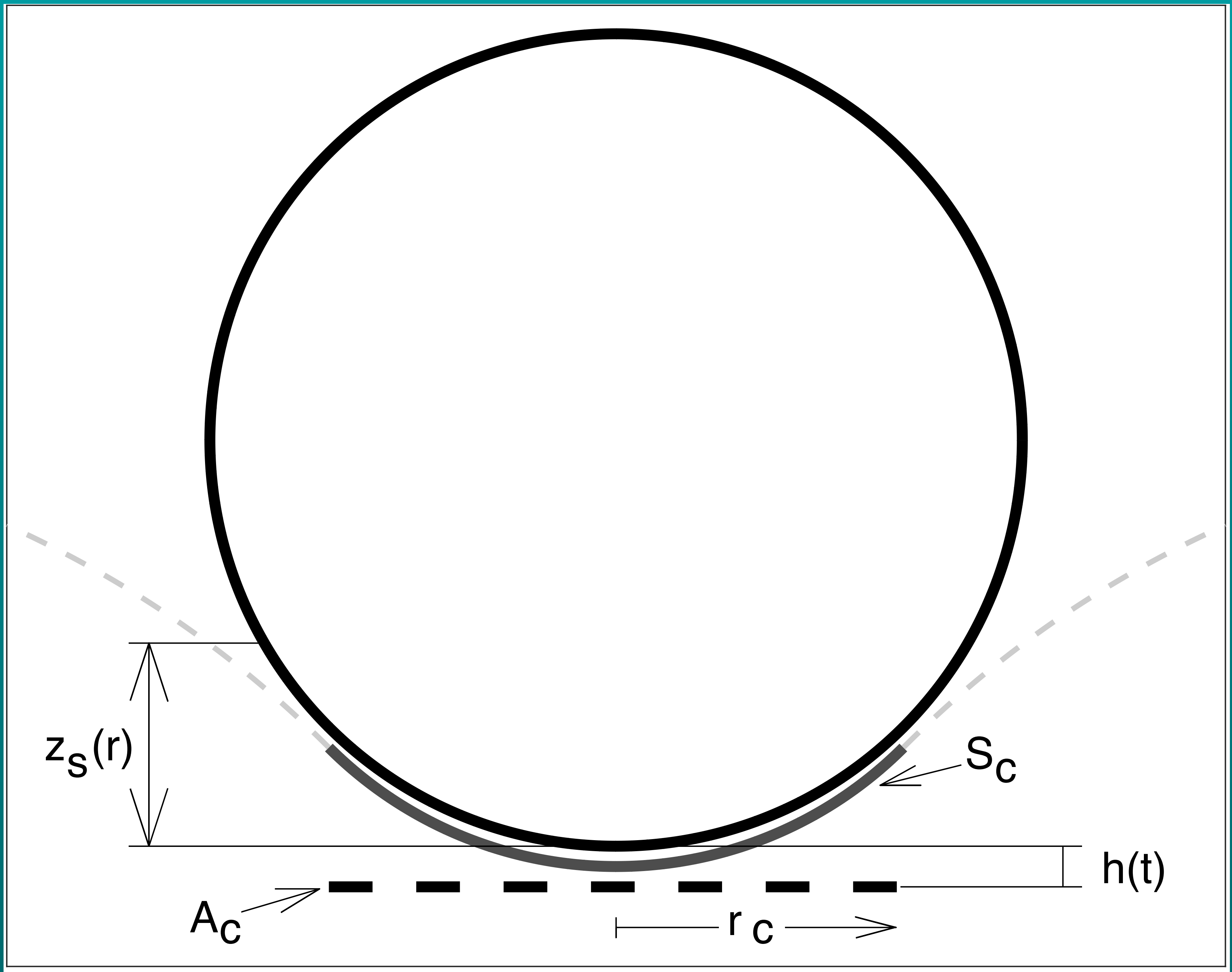
Experimental Verification



The model has good comparison with experiments over a range of dynamics.

Kinematic Match Model

Kinematic match model



Perfectly
Hydrophobic
Rigid
Sphere

0 parameters

Same fluid equations as before but BC on impact surface impose kinematic match.

A Non-local Formulation in Physical Space

$$\Delta\phi = 0, \qquad z \leqslant 0,$$

$$\eta_t = \frac{2}{Re}\Delta_H \eta + \phi_z \qquad z = 0,$$

$$\phi_t = -\frac{1}{Fr}\eta + \frac{1}{We}\,\kappa\left[\eta\right] + \frac{2}{Re}\Delta_H\phi - p_s, \qquad z = 0;$$

subject to $\quad \phi, \nabla\phi \to 0 \quad$ when $\quad \sqrt{x^2+y^2+z^2} \to \infty.$

$$\ldots \int_{\mathbb{R}^2\setminus B(\mathbf{r};\epsilon)} \frac{\phi(\mathbf{r}) - \phi(\mathbf{s})}{|\mathbf{r}-\mathbf{s}|^3} dA(\mathbf{s})$$

$$h_{tt} = -\frac{1}{Fr}h - Dh_t + \frac{1}{M}\int_{r\leq r_c} p_s\, dA$$

$$\eta = h + z_s, \quad \text{where} \quad r \leq r_c;$$

...tion $\quad \eta < h + z_s, \quad \text{where} \quad r_c < r < R_o;$

...urface $\quad p_s = 0, \quad \text{where} \quad r > r_c;$

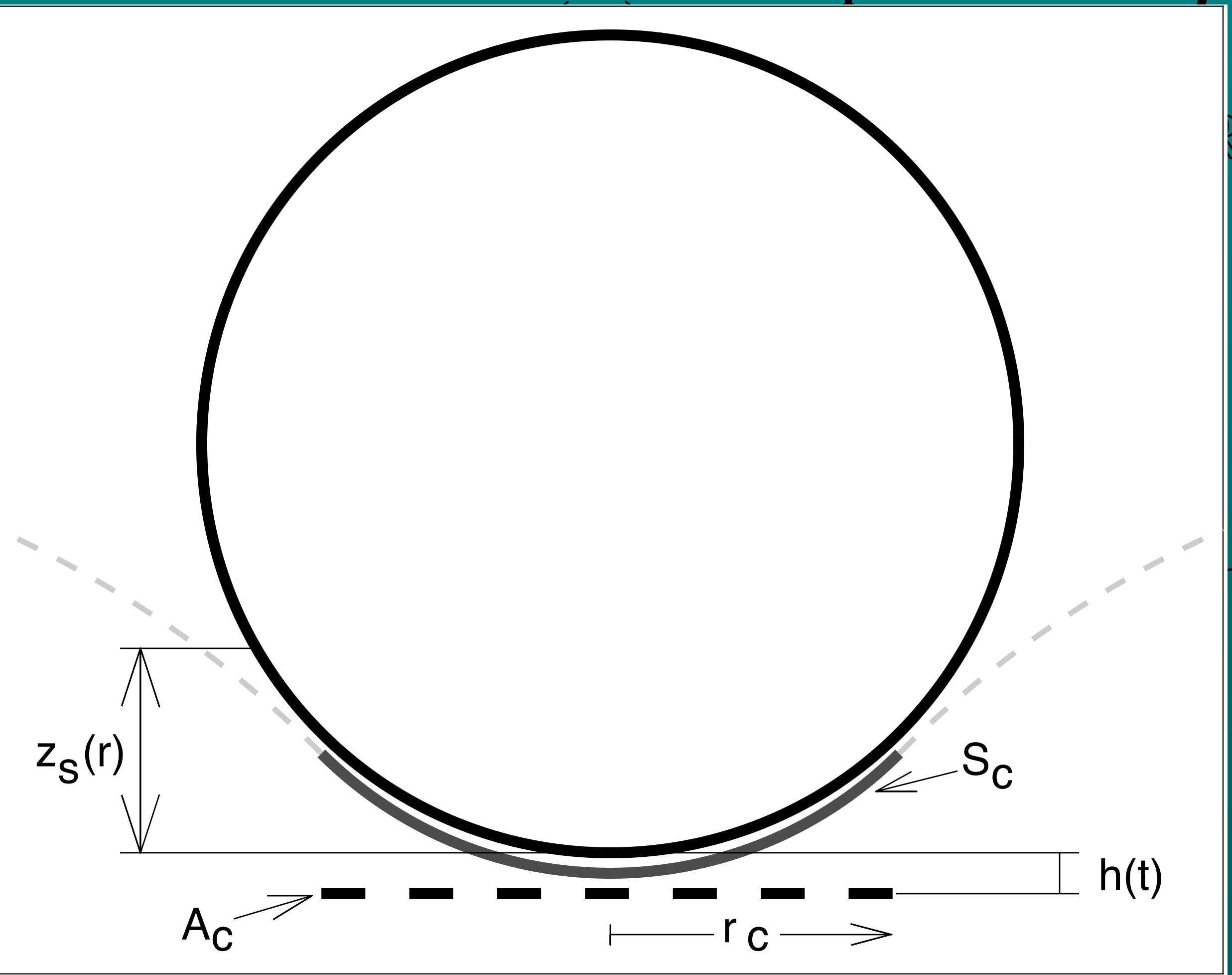
$$\partial_r\eta(r_c) = \partial_r z_s(r_c).$$

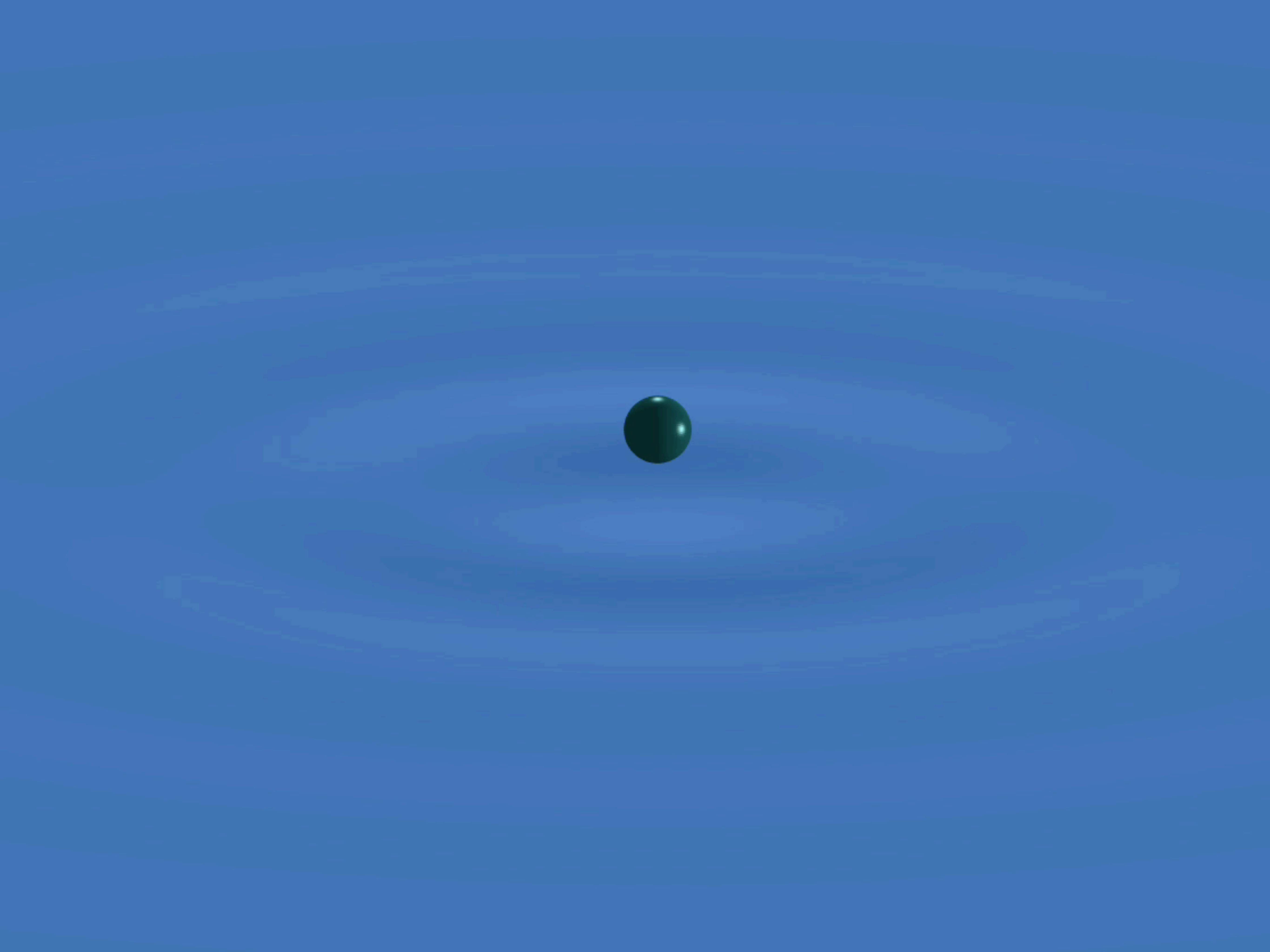
κ = Curvature

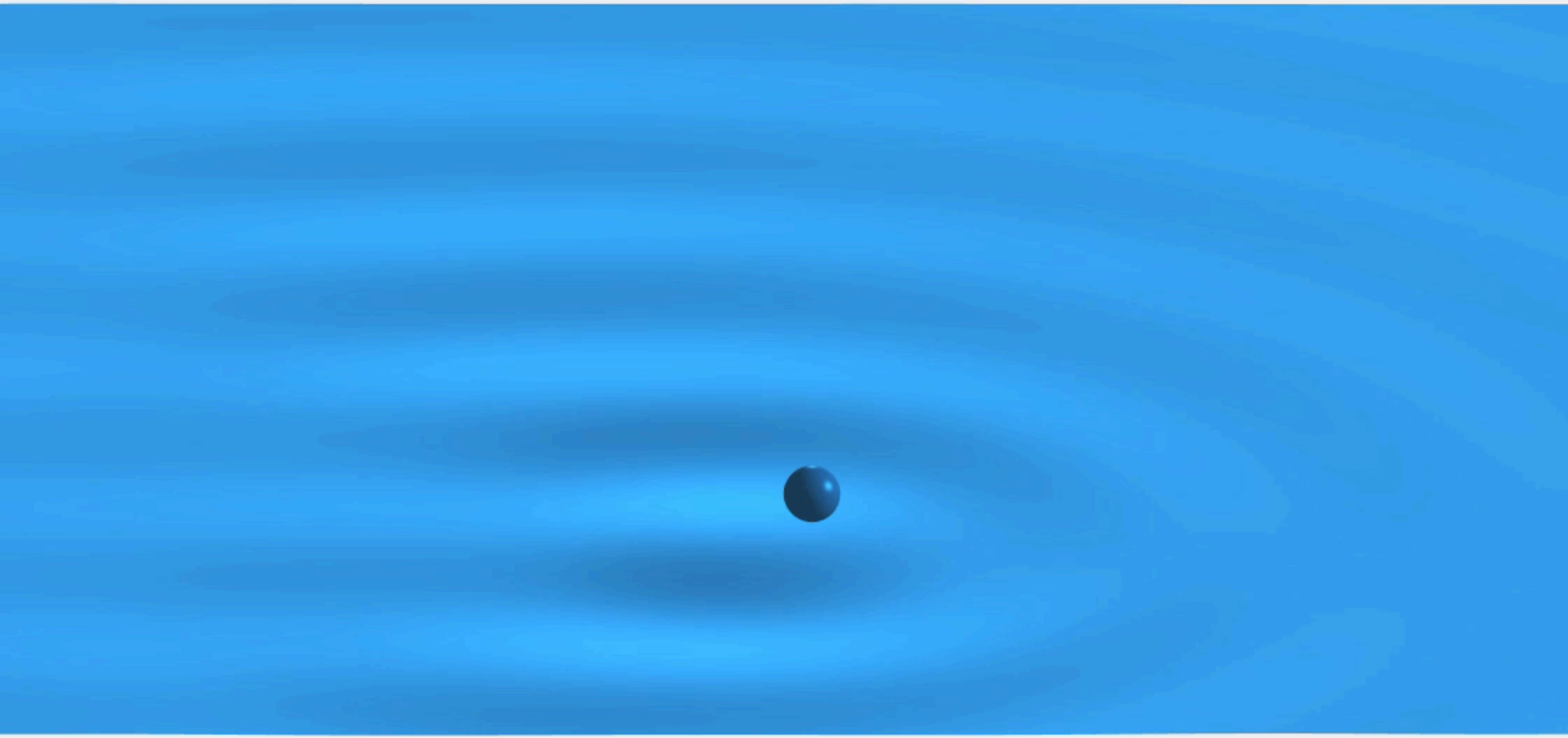
$Re = V_0 R_o/\nu$

$Fr = V_0^2/(gRo)$

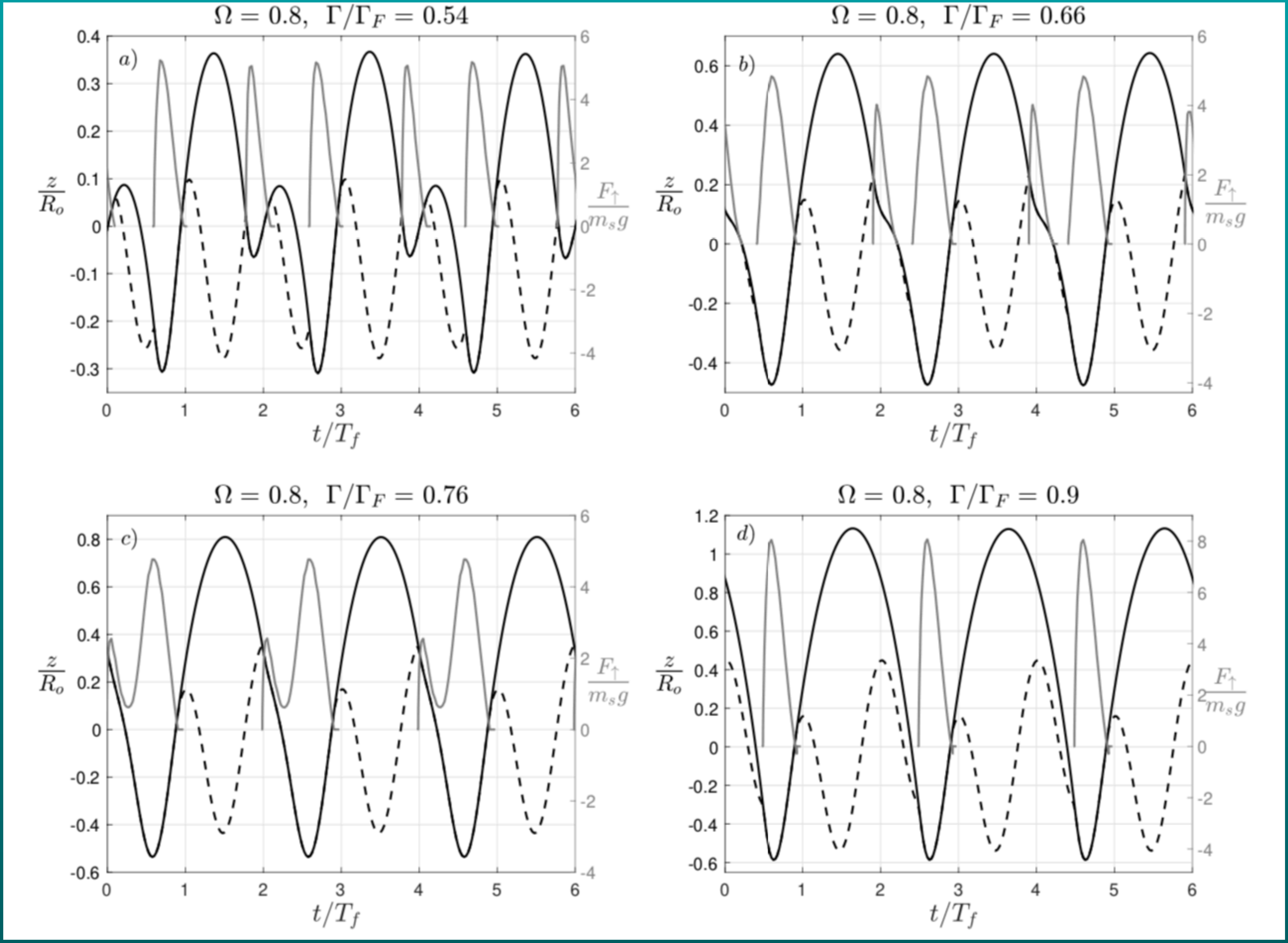
$We = \rho V_0^2 R_o/\sigma$



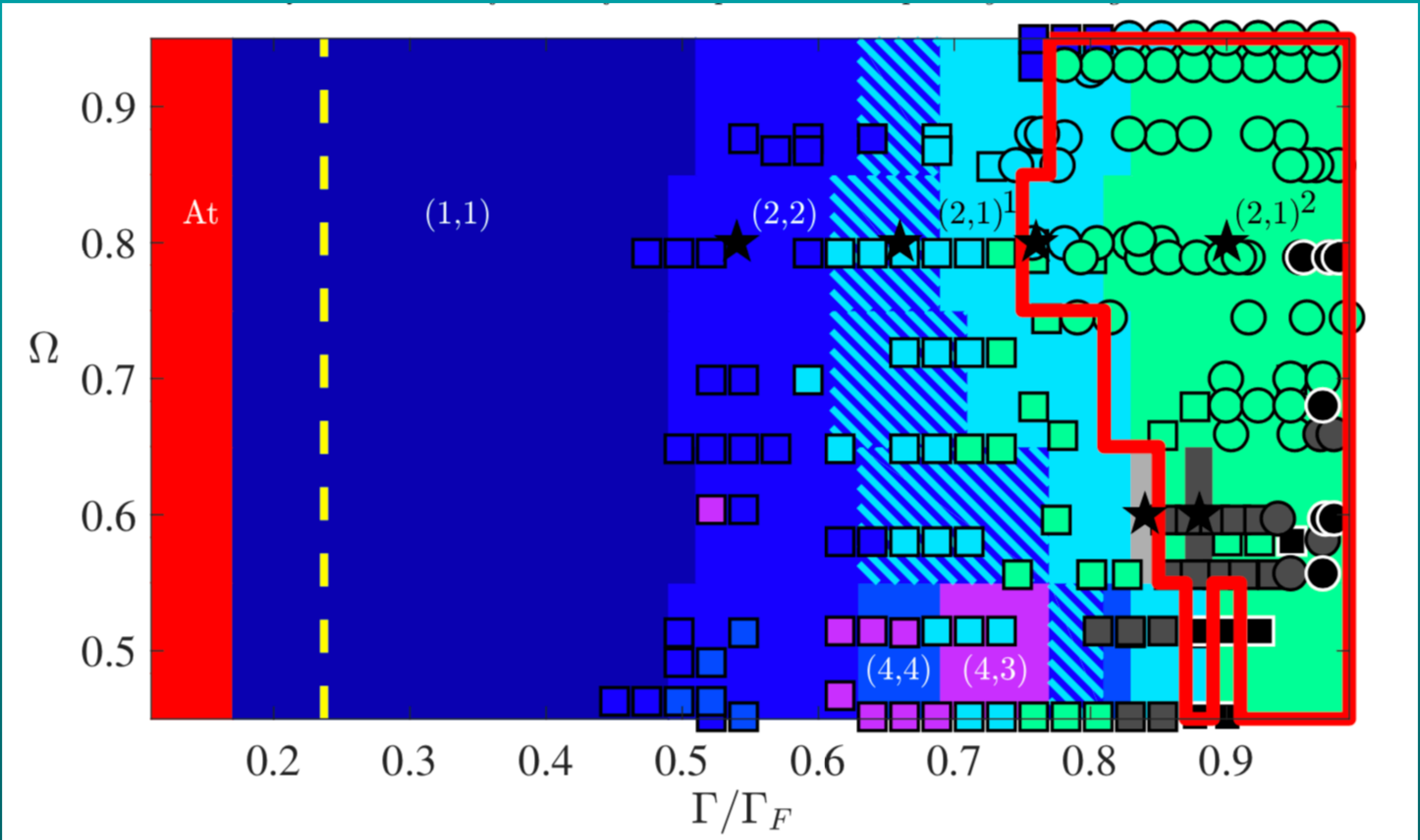




Forces During Impact

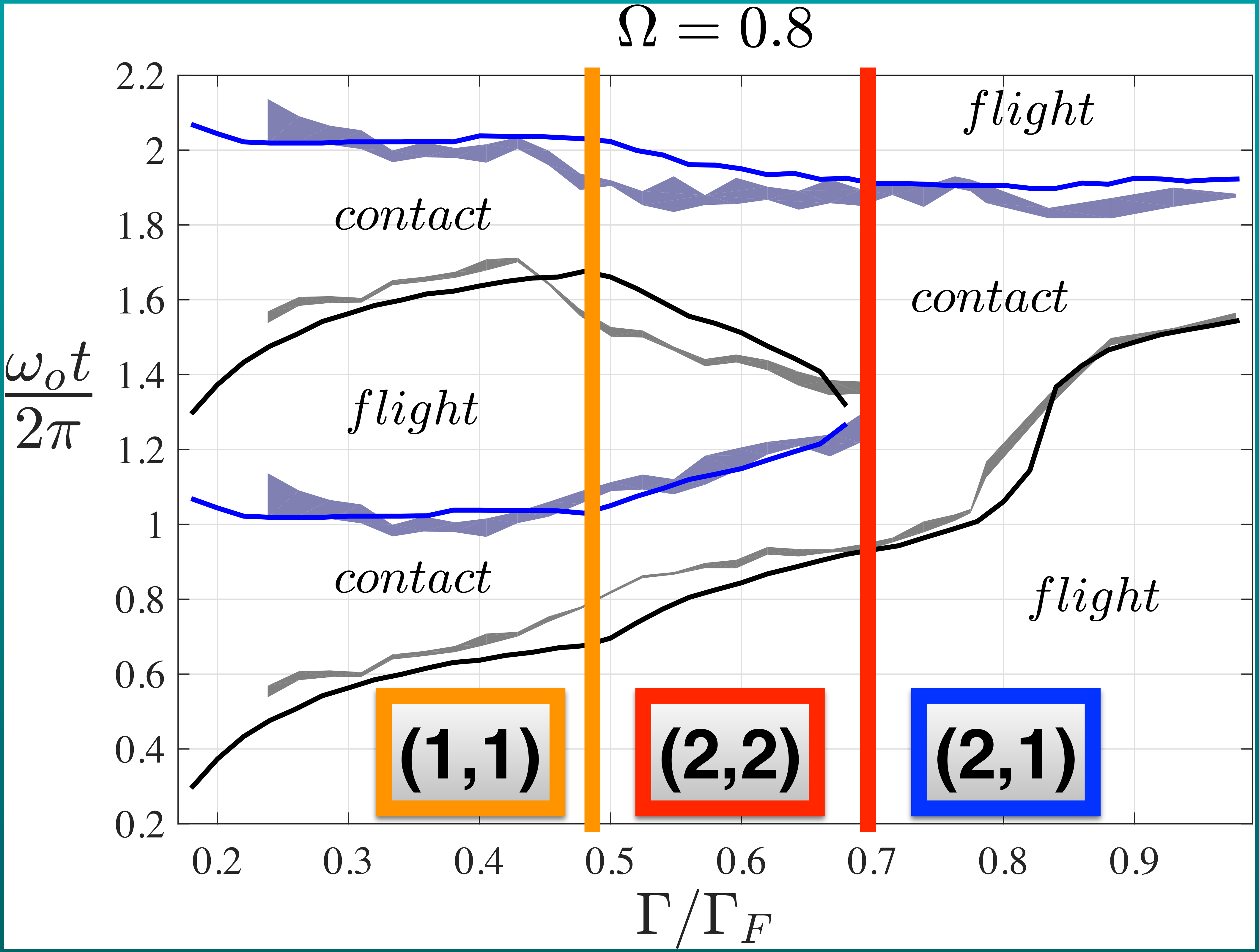


Phase Diagram



Experimental data from: Wind-Willassen, O., Moláček, J., Harris, D.M. & Bush, J. W. M. 2013 Exotic states of bouncing and walking droplets. Phys. Fluids 25, 082002.

Contact Phases



Experimetal data from A. P. Damiano. Surface topography measurements of the bouncing droplet experiment. Master's thesis, Ecole Polytechnique Federale de Lausanne, 2015.

Conclusions

- Macroscopic system of wave-particle association with complex phenomena.
- System mimics qualitatively many quantum phenomena.
- “Virtual laboratory” and mathematical models to explore the behaviour.
- System is nonlinear, chaotic, and particle has path induced memory.

Questions

- Other applications for kinematic match
- Drop deformation
- Hydrodynamics of Faraday pilot-waves in cavities.
- Which equation describes the probability?

References

"Nonlinear Spring" model
(4 parameters)

J. Fluid Mech. (2015), vol. 778, pp. 361–388. © Cambridge University Press 2015 361
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Faraday pilot-wave dynamics: modelling and computation

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Faraday wave–droplet dynamics: discrete-time analysis

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"Kinematic Match" model

Non-wetting impact of a sphere onto a bath and its application to bouncing droplets

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Tunneling with a hydrodynamic pilot-wave model

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Quasi-normal free-surface impacts, capillary rebounds and application to Faraday walkers.

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Dynamics, emergent statistics and the pilot-wave potential of walking droplets

(Chaos)

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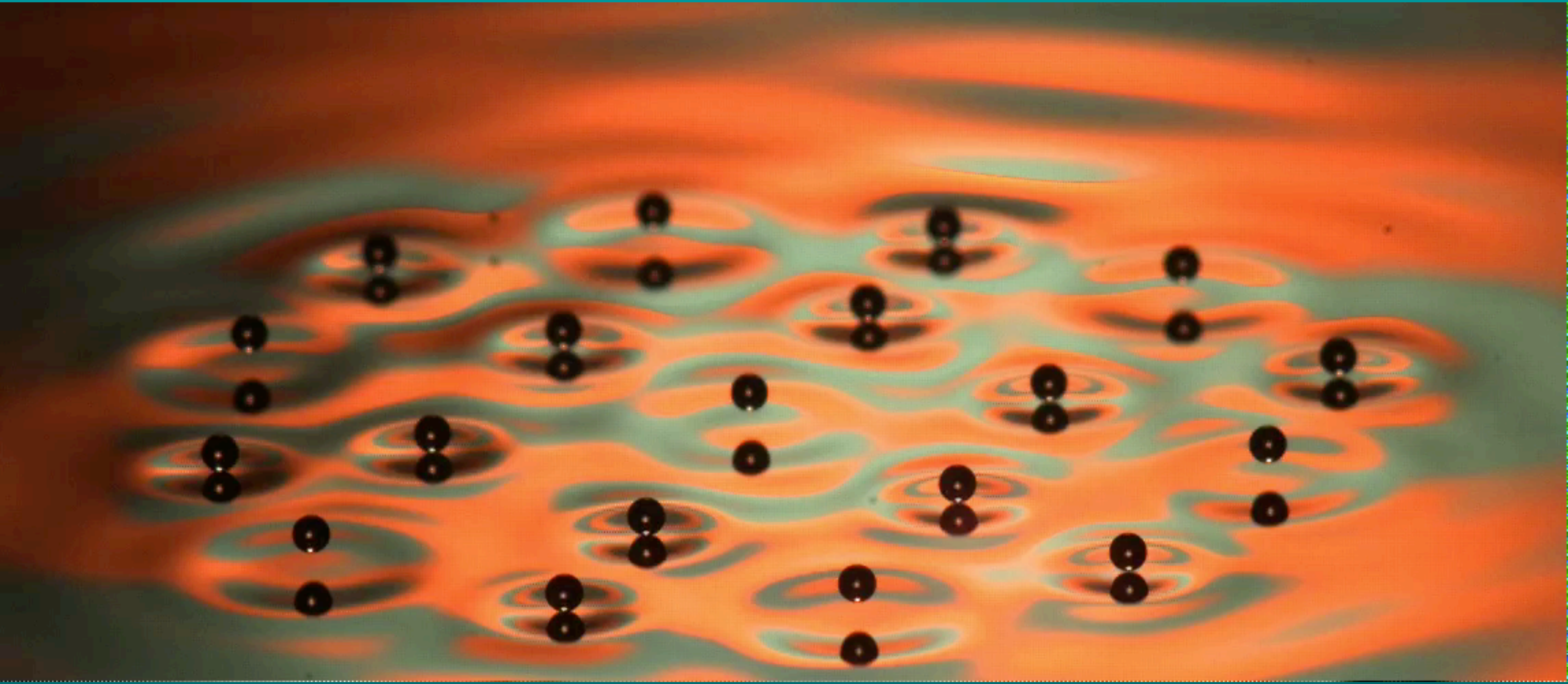

Video: Dan Harris & John Bush

Lattices