Wave-structure impact and rebound at the capillary scale and Faraday pilot waves

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Faraday wave instability

$q = \text{forcing strength}$

$$p = \left( \frac{\omega}{\omega_0/2} \right)^2$$

Benjamin & Ursell 1954

Edwards & Fauve 1994
What happens when a drop falls into a bath?

Videos by John Bush, MIT.
Bouncing droplet on a Faraday stable vibrating bath

Videos by John Bush, MIT.
Walking droplet on vibrating bath: the Faraday Pilot Wave

Videos by John Bush, MIT.
Strobed droplet propelled by its Faraday pilot wave

Videos by John Bush, MIT.
Confinement: particle in a corral

Videos by John Bush, MIT.

\[ \Omega = 50 - 80 \, Hz; \]
HQA: Corral Experiments - particle in a box

Harris et al. Phys. Rev. B 2013
Self-organization into quantized eigenstates of a classical wave-driven particle

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A growing number of dynamical situations involve the coupling of particles or singularities with physical waves. In principle these situations are very far from the wave particle duality at quantum scale where the wave is probabilistic by nature. Yet some dual characteristics were observed in a system where a macroscopic droplet is guided by a pilot wave it generates. Here we investigate the behaviour of these entities when confined in a two-dimensional harmonic potential well. A discrete set of stable orbits is observed, in the shape of successive generalized Cassinian-like curves (circles, ovals, lemniscates, trefoils and so on). Along these specific trajectories, the droplet motion is characterized by a double quantization of the orbit spatial extent and of the angular momentum. We show that these trajectories are intertwined with the dynamical build-up of central wave-field modes. These dual self-organized modes form a basis of eigenstates on which more complex motions are naturally decomposed.

Figure 1 | Principle of the experiment and actual set-up. (a) Sketch of the successive eigenstates of increasing energy and decreasing wavelengths of a quantum particle in a one-dimensional harmonic potential well. (b) These
The Fluid Mechanics Problem
The full problem

The incompressible Navier Stokes

\[ u_t + u \cdot \nabla u = -\frac{1}{\rho} \nabla p + \nu \Delta u + F(t), \quad \nabla \cdot u = 0, \]

\[ F = -g(t) e_z = -g(1 - \Gamma \cos(\omega_0 t)) e_z \]

\( \partial B \) and \( \partial D \) are defined by the equations \( b(x, t) = 0 \) and \( d(x, t) = 0 \)

\[ [\rho \, n - \rho \nu \, \tau \cdot n] = \sigma \kappa n, \quad [u] = 0, \quad D_t b = D_t d = 0. \]

[\cdot] denotes the jump of the quantity across the interface,
\( D_t \) is the material derivative \( \partial_t + u \cdot \nabla \),
\( \sigma \) the surface tension coefficient, \( \kappa \) is the mean curvature of the surface
\( \nu \) is the viscosity, \( \rho \) is the density (with different values in each domain).
\( n \) is the unit normal and \( \tau \) is the strain tensor \( (\nabla u + \nabla u^T) \).
Single bounce of solid hydrophobic sphere

\[ \frac{tV_0}{R_o} = 0.000 \]
CFD (GERRIS) - axisymmetric

Case a) $V_0 = 34.5863$ cm/s

Case f) $V_0 = 61.0775$ cm/s

Case k) $V_0 = 92.2089$ cm/s

Case p) $V_0 = 114.0690$ cm/s
Joint work with Radu Cimpeanu.
Challenges

- Multiscale problem: length scales $\sim 10^{-6} \text{m.}$ to $\sim 10^{-1} \text{m.}$; timescales $\sim 10^{-2} \text{s.}$ to $\sim 10^{3} \text{s.}$
- Free boundary problem: bath surface and droplet deform.
- Modelling is necessary: both for feasibility and to extract the important effects.
- Removing the lubrication layer results in a “nonsmooth” (piecewise smooth) dynamical system.
Modelling Approaches

- **Discrete Waves**: Eddi, Couder, Bush & others: Wavefield as a sum of discrete single Bessel (or simpler) standing waves with time decay.

- **Trajectory Equation**: Rosales, Oza & Bush: Discrete sum approximated by an integral, makes analysis possible.

- **Wave Generation**: M. et. al.: Continuous-time bath-droplet interaction - droplet as a wavemaker - captures further important effects (e.g. Doppler, (m,n) modes, decay).

- **Discrete Wave Generation**: Durey & M.: Discrete-time bath-droplet interaction - allows for fast realistic simulations and analysis.
Discrete Impact Model
(Durey & Milewski 2017)
Simplifying assumptions

Assume:

- Period doubled vertical dynamics (i.e. (2,1) mode).
- Instantaneous impacts: \( f(t) = f_0 \delta(t - t_n) \).
- Impacts occur at a point: \( P(x, t) = f(t)\delta(x - X(t)) \).
- The result: \( f_0 = gT \), where \( T = 4\pi/\omega_0 \) is the time between impacts.

**Fourier-Hankel transform** introduces orthogonal basis functions:

\[
\Phi_m(r, \theta; k) = J_m(kr) \cos(m\theta), \\
\Psi_m(r, \theta; k) = J_m(kr) \sin(m\theta),
\]

for all \( k \in \mathbb{R}^+ \) and for all \( m \in \mathbb{N} \).

Obtain system of **homogeneous ODEs with jump conditions**.
Dynamics and jump conditions

\[ \eta(r, \theta, t) = \sum_{m=0}^{\infty} \int_{0}^{\infty} k \left( a_m(t; k) \Phi_m(r, \theta; k) + b_m(t; k) \Psi_m(r, \theta; k) \right) dk. \]

During flight:

\[ \mathcal{L}_k a_m(t; k) = 0, \quad \mathcal{L}_k b_m(t; k) = 0. \]

\( \mathcal{L}_k \) is a damped Mathieu differential operator.

\[ \mathbf{X}''(t) + V' [\mathbf{X}(t)] = 0. \]

At impacts:

\[ [a'_m(t_n; k)]^\pm = -P_m(k) \Phi_m(\mathbf{X}(t_n); k), \]
\[ [b'_m(t_n; k)]^\pm = -P_m(k) \Psi_m(\mathbf{X}(t_n); k). \]

\[ [\mathbf{X}'(t_n)]^\pm = -F(c) \left( \frac{1}{c} \sqrt{\frac{B}{R}} \nabla \eta(\mathbf{X}(t_n), t_n) + \mathbf{X}'(t_n) \right) \]
Regular walking states - orbits, walkers, pairs, trains

Walking states are found under a *periodicity under shift* condition.

Period Map = Mathieu Map ◦ Graf Addition Map ◦ Jump Conditions Map

- Find discrete-time travelling solutions with speed $\delta x$.
- Linearise map to analyse stability of steady states.
- Lower wave field energy for walking than unstable bouncing.
- Wave field has exponential spatial decay and Doppler shift.

Figure: Left: full solution (black) with analytical approximation (grey) valid for $\delta x \ll 1$. Right: Wave field for $\delta x = 0.08$ and $\Gamma/\Gamma_F = 0.96$. 

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Modelling and computation of droplet - Faraday pilot wave interaction.

Walking dynamics ($\mathbf{t} = 0$) Find discrete-time travelling solutions with speed $\delta x$.

Linearise map to analyse stability of steady states.

Lower wave field energy for walking than unstable bouncing.

Wave field has exponential spatial decay and Doppler shift.

Matt Durey Faraday wave-droplet dynamics
Two orbiting droplets - quantization

- In-phase and out of phase orbiters obtained exactly. Map composition also gives stability.
- Obtain stable, weakly unstable (Hopf), strongly unstable solutions.
Exotic orbits

- Circular orbits destabilize to more exotic orbits (lemniscate, trefoil, butterfly).
- Experimentally observed double quantization in $\bar{R}$ and $\bar{L}_z$ (Perrard et al 2014).
Simulate long trajectory starting from (unstable) circular orbit. Segment trajectory at points of maximum radius and compute mean radius $\bar{R}$ and angular momentum $\bar{L}_z$ over each sub-trajectory. Compute cluster centroids using $K$-means clustering.

Double Quantization in $\bar{R}$ and $\bar{L}_z$!
- Effective potential \( \equiv \) mean wave field + harmonic potential.
- Gradient of effective potential corresponds to directions of mesh.

\[ \tilde{\eta}(x) = \int_{\mathbb{R}^2} \eta_B(x - y)\mu(y)\,dy = (\eta_B * \mu)(x), \]  

\textbf{Theorem 1.} The stationary probability distribution \( \mu(x) \) for the droplet position and the mean wave field at impact \( \tilde{\eta}(x) \) are related by

where \( \eta_B(x) \) is the radially symmetric wave field of a bouncer centred at the origin.\(^1\)
Continuous-time wave generation and nonlinear spring model for bouncing
Modelling the impact

Consider the droplet to be centred at \((X(t), Z(t))\), and a free surface \(\eta(x, t)\).

\[
    m\ddot{Z} = -mg(t) + \left\{ -\alpha_f \dot{Z} \right\} l_f(t) + \mathcal{F} \left( Z, \dot{Z}, \eta, \eta_t \right) l_i(t),
\]

\[
    m\ddot{X} = \left\{ -\alpha_f \dot{X} \right\} l_f(t) + \left\{ -\alpha_i \dot{X} - \nabla \tilde{\eta}|_{x=X} \right\} \mathcal{F} l_i(t),
\]

- \(l_f\) and \(l_i\) are indicator functions of flight and impact
- \(\mathcal{F} \left( Z, \dot{Z}, \tilde{\eta}, \eta_t \right)\) is the force normal to the free surface exerted on the droplet.
- \(m\) is the droplet mass, \(\alpha_f\) and \(\alpha_i\) are flight and impact drag coefficients
- Replacing \(\eta\) by \(\tilde{\eta}\) in \(\mathcal{F}\), an average representation of the bath’s free surface (notion of a “penetration depth” \(h(t) = |Z - R - \tilde{\eta}|\))
Modelling - the waves

Linearised quasi-potential approximation for weakly damped waves

\[ \Delta \phi = 0, \quad z < 0, \]
\[ \phi_t = -g(t) \eta + \frac{\sigma}{\rho} \Delta_H \eta + 2\nu \Delta_H \phi - \frac{1}{\rho} P_D(x - X(t), t), \quad z = 0, \]
\[ \eta_t = \phi_z + 2\nu \Delta_H \eta, \quad z = 0, \]
\[ P_D = \mathcal{F}(t) I(|x - X| < R(t))/\pi R(t)^2 \]

- \( \phi \) is the potential in a Helmholtz decomposition \( u = \nabla \phi + \nabla \times \Psi(\phi) \).
- \( R(t) \) is a contact radius - modelled geometrically \( R(Z - \bar{\eta}) \).
- \( \Delta_H \) is the horizontal Laplacian.
Modelling the impact (cont)

- $l_i(t)$ must be calculated from geometric information on the waves and flight of drop.
- The dynamical system is now non-smooth: there is a switch between impact and flight.

One possible vertical dynamics (ie a model for $\mathcal{F}$) (see Molacek and Bush 2014) is:

\[
m\ddot{Z} = -mg(t),
\]

\[
\left(1 + \frac{c_3}{\ln^2 \left| \frac{c_1 R_0}{Z-\eta} \right|} \right) m\ddot{Z} + \frac{4}{3} \frac{\pi \mu R_0 c_2}{\ln \left| \frac{c_1 R_0}{Z-\eta} \right|} \left( \dot{Z} - \dot{\eta} \right) + \frac{2\pi \sigma}{\ln \left| \frac{c_1 R_0}{Z-\eta} \right|} (Z - \eta) = -mg(t),
\]

during flight and impact respectively.
Complex bouncing and walking

\[(m, n)^p\] states: \(m\) is number of forcing periods, \(n\) is number of bounces, \(p\) is an energy state. The control parameter is \(\Gamma\).
Experimental Verification

The model has good comparison with experiments over a range of dynamics.
Kinematic Match Model
Non-wetting impact of a sphere onto a bath and bouncing droplets

Figure 1. Schematics of an axial section of the hydrophobic impact of a solid sphere onto a free fluid surface. The unpressed free surface is shown in the dashed light grey line, the contacted portion of the fluid interface $S_C$ is shown in dark grey. The dashed line sits on the level of the undisturbed free surface ($z = 0$), and its length corresponds to the diameter of $A_C$ (i.e. $2r_c$), the normal projection of the contacted spherical cap $S_C$ on the horizontal plane.

We write equation (2.26) for $h$ and in dimensionless form and obtain

$$h_{tt} = \frac{1}{2} Fr Dh_t + \frac{1}{M} Z A_C p_s dA,$$

(2.27)

where $D = c_f L / (m V^2)$ and $M = m / \rho L^3$.

We assume that the bottom part of the solid body is described by a function $z_s(r)$ with smooth curvature in the vicinity of the impacting region. We recall that the problem is axisymmetric, and thus we define a radial coordinate system given by $r = x^2 + y^2$.

Since functions of $(x, y)$ are simply functions of $r$, the main constraint for the fluid-solid interaction can then be stated as

$$\gamma(r, t) \leq h(t) + z_s(r),$$

(2.28)

which must hold everywhere under the solid, and where we assume $z_s(0) = 0$.

We impose a second natural constraint, namely that the contact angle, at the boundary of the pressed surface $S_C$ (see figure 1), has to be $\pi$. This assumption is equivalent to stating that the effect of surface tension at the boundary of $S_C$ is exactly equal to the effect of the jump in pressure due to the curvature of $S_C$. This can be seen using the method presented in Keller (1998), with the difference that integrations in this case need to be carried out in the contact area and not outside of it.

We ignore the dynamics of air and thus identify $A_C$ as the region of the $z = 0$ plane where relation (2.28) is satisfied as an equality. In practice, this means that there will be no distinction between the pressed part of the free surface and that of the solid. Figure 1 might induce the reader to think that there in fact exists a difference in height between the two contacted surface portions, however this is not the case in the model. The separation shown in figure 1 serves merely didactic purposes.

We thus wish to solve equations (2.24) and (2.27) subject to conditions (2.25) and (2.28). Based on our symmetry and convexity assumptions, we have that $A_C$ must be a disc. An important particularity of the system given by (2.24) and (2.27) is that we Kinematic match model

Same fluid equations as before but BC on impact surface impose kinematic match.
A Non-local Formulation in Physical Space

\[ \Delta \phi = 0, \]
\[ \eta_t = \frac{2}{Re} \Delta_H \eta + \phi_z \]
\[ \phi_t = -\frac{1}{Fr} \eta + \frac{1}{Ve} \kappa \eta + \frac{2}{Re} \Delta_H \phi - p_s, \]

subject to \( \phi, \nabla \phi \rightarrow 0 \) when \( \sqrt{x^2 + y^2 + z^2} \rightarrow \infty \).

\[ \eta = h + z_s, \]
\[ \kappa = \text{Curvature}, \]
\[ \eta < h + R_s, \]
\[ p_s = 0, \]
\[ \partial_r \eta(r_c) = \partial_r z_s(r_c) = \rho V_0^2 R_o / \sigma \]
Forces During Impact

\[ \Omega = 0.8, \quad \Gamma / \Gamma_F = 0.54 \]

\[ \Omega = 0.8, \quad \Gamma / \Gamma_F = 0.66 \]

\[ \Omega = 0.8, \quad \Gamma / \Gamma_F = 0.76 \]

\[ \Omega = 0.8, \quad \Gamma / \Gamma_F = 0.9 \]
Contact Phases

\[ \Omega = 0.8 \]

Conclusions

• Macroscopic system of wave-particle association with complex phenomena.
• System mimics qualitatively many quantum phenomena.
• “Virtual laboratory” and mathematical models to explore the behaviour.
• System is nonlinear, chaotic, and particle has path induced memory.

Questions

• Other applications for kinematic match
• Drop deformation
• Hydrodynamics of Faraday pilot-waves in cavities.
• Which equation describes the probability?
Non-wetting impact of a sphere onto a bath and its application to bouncing droplets

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Nonlinear Spring” model
(4 parameters)

Faraday pilot-wave dynamics: modelling and computation

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“Kinematic Match” model

Non-wetting impact of a sphere onto a bath and its application to bouncing droplets

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Quasi-normal free-surface impacts, capillary rebounds and application to Faraday walkers.

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Dynamics, emergent statistics and the pilot-wave potential of walking droplets

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Faraday wave–droplet dynamics: discrete-time analysis

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Video: Dan Harris & John Bush
Lattices