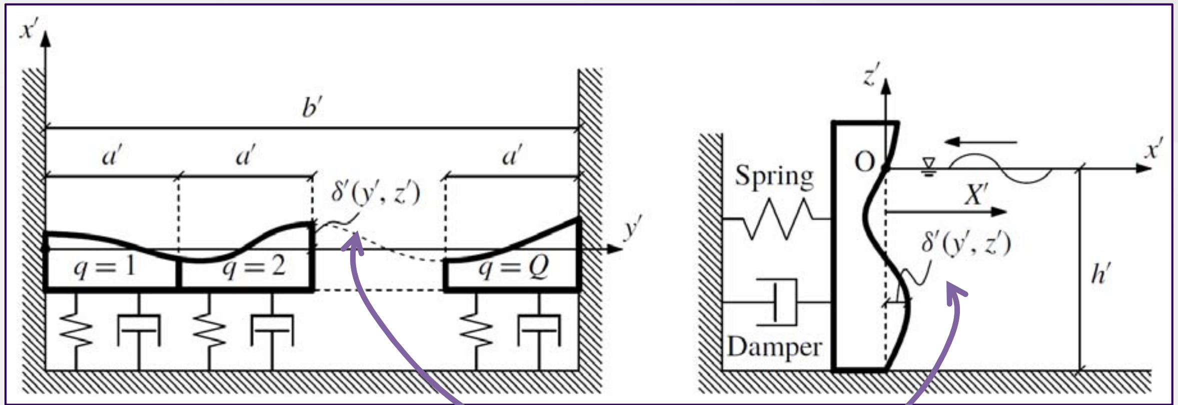
# Weakly nonlinear theory of an array of curved wave energy converters

S. Michele

*Department of Mathematical Sciences, Loughborough University (UK)*

# Today you will see…




… how we can change the shape of an Oscillating Wave Surge Converter (OWSC) to maximize energy production.

The gate model is similar to that of Carrier (1970) except for a weak horizontal deviation of the gate wetted surface about the vertical plane.

# State of the art

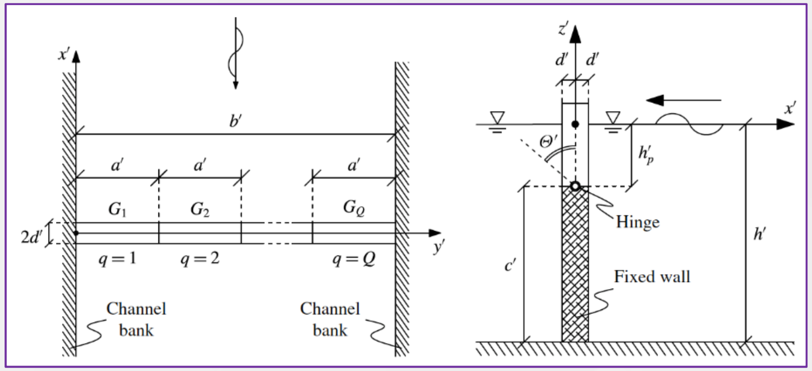
- The hydrodynamics of flaps has been extensively analysed in the past. Hereafter few works:

- Plates
  - Parsons & Martin (1992, 1994, 1995), Evans & Porter (1996), Dean & Dalrymple (1991), Mei *et al.* (2005), Falnes (2007), Linton & McIver (2001), Porter (2014).
- Venice gates
  - Li & Mei (2003), Mei *et al.* (1994), Adamo & Mei (2005).
- OWSCs in a channel
  - Renzi & Dias (2012, 2013), Sammarco *et al.* (2013).
- OWSCs in open sea
  - Renzi & Dias (2013, 2014), Michele *et al.* (2015, 2016), Noad & Porter (2015).
- Nonlinear theories
  - Sammarco *et al.* (1997), Vittori *et al.* (1996).

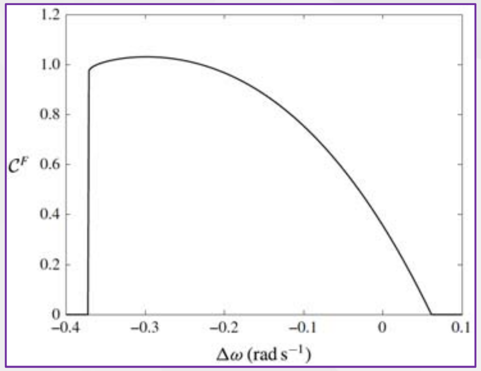
- And so on…

# Motivation of the study

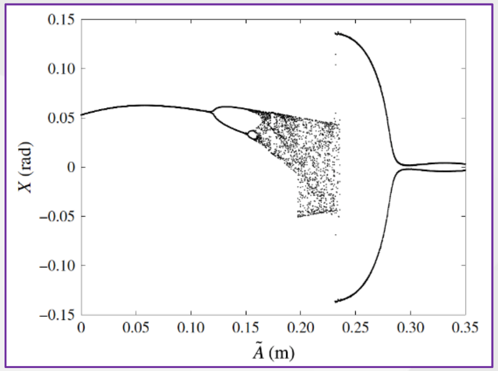
- Most wave theories applied to OWSCs neglect **nonlinear** hydrodynamic-related terms
- Recently, Michele *et al.* (2018) showed that occurrence of **subharmonic resonance** and **mode competition** of trapped modes increases efficiency




Plan geometry and side view of the array



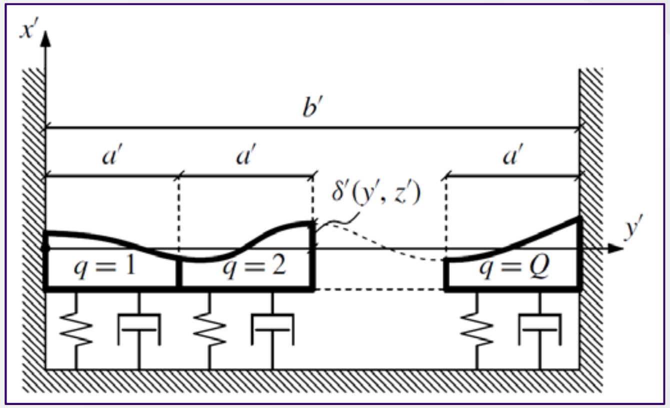

Capture Factor vs Detuning incoming uniform waves

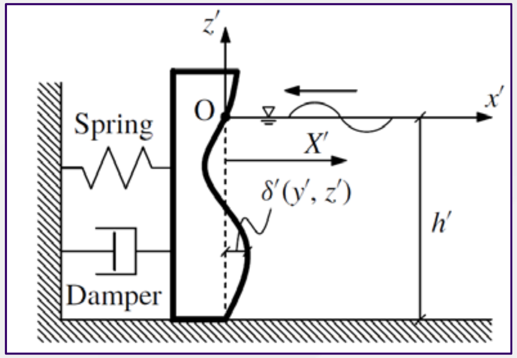



Period doubling scenarios in modulated incident waves

- Analytical solutions exist for simple geometries (rectangular, circular, elliptical,…)
- In this study we analyse the effects of more **complex geometries** in **nonlinear regimes**

# Mathematical model




Gate surface equation

$$x' - X'(y', t') - \delta'(y', z') = 0$$

Non-dimensional quantities

$$(x, y, z) = (x', y', z')/\lambda', \quad \Phi = \Phi'/(A'_T \omega' \lambda'), \quad \zeta = \zeta'/A'_T, \quad t = t'\omega',$$
$$(a, b, h) = (a', b', h')/\lambda', \quad X = X'/A'_T, \quad \delta = \delta'/\delta'_g, \quad G = g'/(\omega'^2 \lambda'),$$

Two small parameters

$$\epsilon = A'_T/\lambda' \ll 1, \quad \mu = \delta'_g/\lambda' \ll 1, \quad \mu = O(\epsilon).$$

Governing equation

$$\nabla^2 \Phi = 0$$

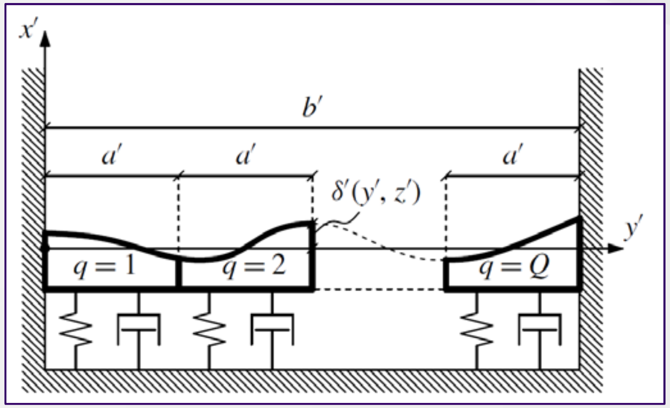
Boundary conditions on the free surface

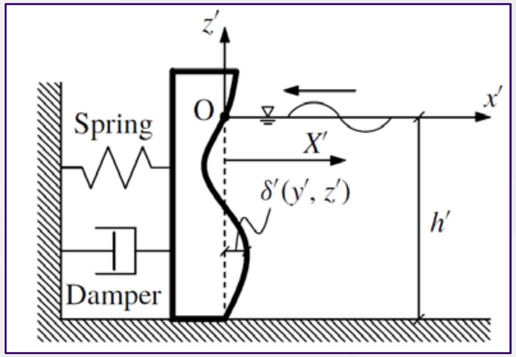
$$-G\zeta = \Phi_t + \epsilon \tfrac{1}{2} |\nabla \Phi|^2, \quad z = \epsilon\zeta,$$
$$\Phi_{tt} + G\Phi_z + \epsilon |\nabla \Phi|_t^2 + \epsilon^2 \tfrac{1}{2} \nabla \Phi \cdot \nabla |\nabla \Phi|^2 = 0, \quad z = \epsilon\zeta,$$

No-flux boundary conditions

$$\Phi_z = 0, \quad z = -h,$$
$$\Phi_y = 0, \quad y = 0 \quad \text{and} \quad y = b.$$

# Mathematical model




Kinematic condition on the array surface

$$\Phi_x = X_t + \mu(\Phi_y\delta_y + \Phi_z\delta_z)$$

$$x = \epsilon X + \mu\delta$$

Equation of motion of the $qth$ gate coupled with a linear damper

$$\epsilon M X_{q,tt} + \epsilon C G X_q + \epsilon^3 \nu X_{q,t} = \int_{(q-1)a}^{qa} \mathrm{d}y \left\{ \int_{-1}^{\epsilon\zeta} \mathrm{d}z \left( \epsilon\Phi_t + \epsilon^2 \frac{1}{2} |\nabla\Phi|^2 \right) + \int_0^{\epsilon\zeta} Gz\, \mathrm{d}z \right\}$$

$M = M'/(\rho'\lambda'^3)$ Non-dimensional mass

$C = C'/(g'\rho'\lambda'^2)$ Non-dimensional stiffness

$\nu = \nu'/(A_T'^2\omega'\rho'\lambda')$ Non-dimensional PTO coefficient

# Multiple-scale analysis and three timing

Let us introduce the following expansion for the unknowns

$$\Phi = \Phi_1(x, y, z, t, t_1, t_2) + \epsilon\Phi_2(x, y, z, t, t_1, t_2) + \epsilon^2\Phi_3(x, y, z, t, t_1, t_2) + O(\epsilon^3),$$
$$\zeta = \zeta_1(x, y, t, t_1, t_2) + \epsilon\zeta_2(x, y, t, t_1, t_2) + \epsilon^2\zeta_3(x, y, t, t_1, t_2) + O(\epsilon^3),$$
$$X_q = X_{q,1}(t, t_1, t_2) + \epsilon X_{q,2}(t, t_1, t_2) + \epsilon^2 X_{q,3}(t, t_1, t_2) + O(\epsilon^3),$$
$$X = X_1(y, t, t_1, t_2) + \epsilon X_2(y, t, t_1, t_2) + \epsilon^2 X_3(y, t, t_1, t_2) + O(\epsilon^3),$$

Three-timing is necessary to avoid ***secularity*** at the second and third order

$$t_1 = \epsilon t \qquad t_2 = \epsilon^2 t$$

Higher order solutions imply higher harmonics. Return in physical variables and assume the following expansion

$$\{\Phi_n, \zeta_n, X_{q,n}, X_n\} = \sum_{m=0}^{n}\{\phi_{nm}, \eta_{nm}, \chi_{q,nm}, \chi_{nm}\}e^{-im\omega t} + *$$

# Multiple-scale analysis and three timing

$$\nabla^2 \phi_{nm} = 0, \quad \text{in } \Omega,$$
$$\phi_{nm_z} = \phi_{nm} \frac{m^2\omega^2}{g} + \mathcal{F}_{nm}, \quad z = 0,$$
$$\eta_{nm} = \phi_{nm} \frac{\mathrm{i}m\omega}{g} + \mathcal{B}_{nm}, \quad z = 0,$$
$$\phi_{nm_z} = 0, \quad z = -h,$$
$$\phi_{nm_y} = 0, \quad y = 0, y = b,$$
$$\phi_{nm_x} = -\mathrm{i}m\omega\chi_{nm} + \mathcal{G}_{nm}, \quad x = 0.$$

$$-m^2\omega^2 M \chi_{q,nm} + C\chi_{q,nm} = -\mathrm{i}m\omega\rho \int_{(q-1)a}^{qa} \mathrm{d}y \int_{-h}^{0} \phi_{nm}\, \mathrm{d}z + \mathcal{D}_{nm}$$

The nonlinear set of governing equations and boundary conditions is decomposed in a sequence of linear boundary-value problems of order $n$ and harmonic $m$

The forcing terms $F_{nm}$, $B_{nm}$ , $G_{nm}$ and $D_{nm}$ are defined for each order.

# Leading order solution *O(1)*

Zero-th harmonic unforced with homogeneous b.c. →

$$\phi_{10} = \phi_{10}(t_1, t_2), \quad \chi_{10} = 0, \quad \eta_{10} = 0.$$

First harmonic yields the ***trapped mode*** solution

$$\phi_{11} = \mathrm{i}\chi\omega \sum_{q=1}^{Q} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} \frac{b_{mq} D_n}{C_n \alpha_{nm}} \mathrm{e}^{-\alpha_{nm} x} \cos \frac{m\pi y}{b} \cosh k_n (h+z) \equiv \mathrm{i}\chi f_{11}(x, y, z)$$

Dispersion relation

$$\left.\begin{aligned} \omega^2 &= g k_0 \tanh k_0 h, \\ \omega^2 &= -g \bar{k}_n \tan \bar{k}_n h, \quad k_n = \mathrm{i}\bar{k}_n, \; n = 1, \ldots, \infty. \end{aligned}\right\}$$

Real coefficients

$$b_{mq} = r_q \frac{2}{m\pi} \left[ \sin \frac{qm\pi}{Q} - \sin \frac{(q-1)m\pi}{Q} \right],$$

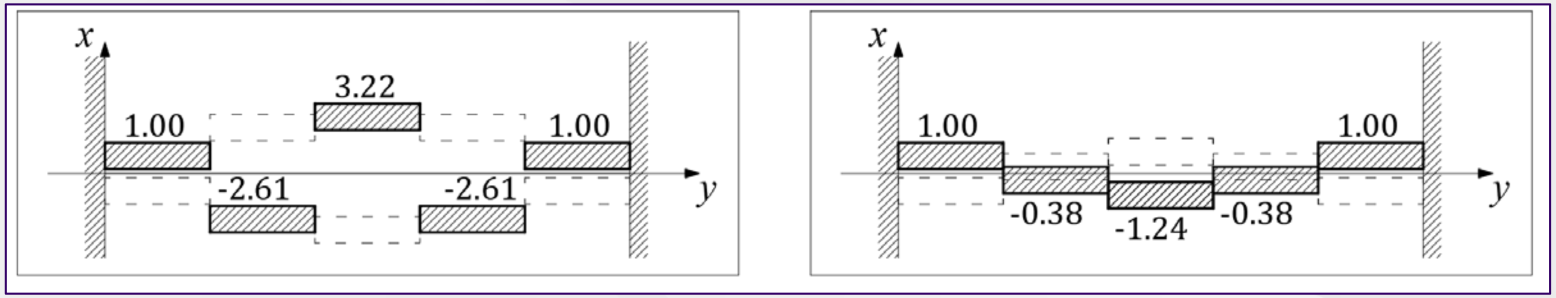
$$\alpha_{nm} = \sqrt{\left(\frac{m\pi}{b}\right)^2 - k_n^2}, \quad C_n = \frac{1}{2}\left(h + \frac{g}{\omega^2} \sinh^2 k_n h\right), \quad D_n = \frac{\sinh k_n h}{k_n}.$$

Solution of the equation of moton gives (*Q-1*) **out-of-phase** natural trapped modes and related eigenfrequencies

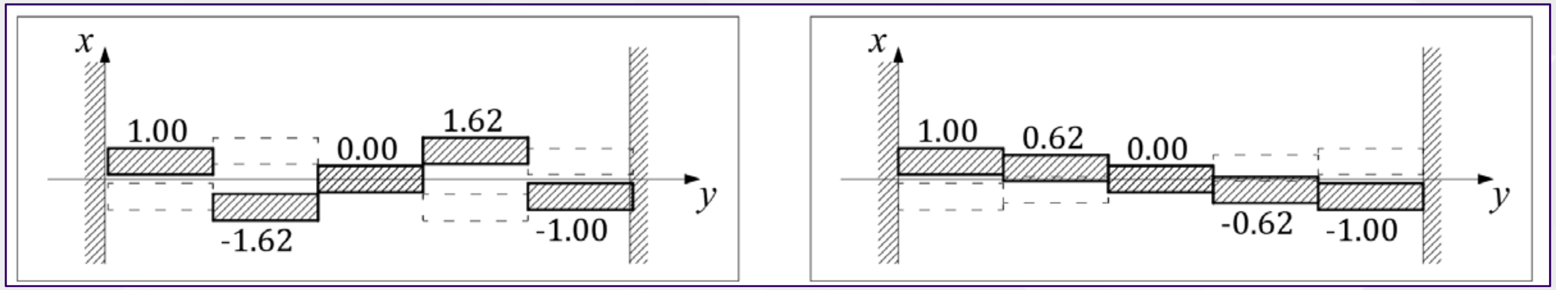
# Leading order solution *O(1)*

- Examples of *(Q-1)* trapped modes for an array of *Q=5* gates
- The number over each gate represents the normalized amplitude

- Even modes




- Odd modes



## Second order problem $O(\varepsilon)$: zero-th harmonic

Forcing terms on the free surface and on the gate surface yield a second order drift

$$\eta_{20} = -\frac{\mathrm{i}}{g\epsilon}[\phi_{10_{t_1}}\epsilon\omega + |\chi|^2(f_{11_x}^2 + f_{11_y}^2 - f_{11_z}^2)],$$

$$\chi_{q,20} = \frac{\rho}{C\epsilon}\int_{(q-1)a}^{qa} \mathrm{d}y \left\{ -\frac{\omega^2|\chi|^2}{g} f_{11}^2 \Big|_{z=0} + \int_{-h}^{0} \mathrm{d}z\,[\phi_{10_{t_1}}\epsilon\omega + |\chi|^2(-f_{11_x}^2 + f_{11_y}^2 + f_{11_z}^2)] \right\}_{x=0}$$

→ Bound wave

→ Static displacement

- No dependence on the gate shape $\delta$
- $X_{20}$ does not affect power extraction

## Second order problem $O(\varepsilon)$: first harmonic

The gate shape 'forces' the first harmonic problem

Solvability condition

$$\iiint_{\Omega} (\phi_{11}\nabla^2\phi_{21} - \phi_{21}\nabla^2\phi_{11})\,\mathrm{d}\Omega = \iint_{\partial\Omega} \left(\phi_{11}\frac{\partial\phi_{21}}{\partial n} - \phi_{21}\frac{\partial\phi_{11}}{\partial n}\right)\mathrm{d}S = 0$$

# Second order problem $O(\varepsilon)$: first harmonic – the gate shape

Evolution equation

$$\chi_{t_1} - \frac{\mathrm{i}c_\delta}{\omega\epsilon}\chi = 0$$

$$c_\delta = \frac{1}{c_f}\int_0^b \mathrm{d}y \int_{-h}^0 \mathrm{d}z\{f_{11}(f_{11_{xx}}\delta - f_{11_y}\delta_y - f_{11_z}\delta_z) + r\omega\delta f_{11_x}\}$$

$\delta=0$ (flat gate) implies $c_\delta=0$ thus $X$ depends on the slow time scale $t_2$ only

$$c_f = \int_0^b \mathrm{d}y \int_0^{+\infty} \frac{2\omega f_{11}^2}{g}\,\mathrm{d}x + \int_0^b \mathrm{d}y \int_{-h}^0 2f_{11}r\,\mathrm{d}z + \sum_{q=1}^{Q} \frac{2a\omega M r_q^2}{\rho}$$

Solution

$$\chi(t_1, t_2) = \vartheta(t_2)\mathrm{e}^{-(\mathrm{i}c_\delta t_1/\omega\epsilon)} = \vartheta(t_2)\mathrm{e}^{-\mathrm{i}c_\delta t}$$

→ $c_\delta$ represents a modulation of the modal amplitude growth

# Response to synchronous incident waves

This is a diffraction – radiation problem forced by the incident wave field. The gates move at unison in phase.

Linearity allows the following decomposition →

$$\phi^{A^{\pm}} = \phi^{I} + \phi^{S} + \phi^{R^{\pm}}$$

Velocity potential incident waves of amplitude $A$

$$\phi^{I} = -\frac{\mathrm{i}Ag}{2\epsilon\omega}\frac{\cosh k_0(h+z)}{\cosh k_0 h}\mathrm{e}^{-\mathrm{i}k_0 x}$$

Scattered wave potential

$$\phi^{S} = -\frac{\mathrm{i}Ag}{2\epsilon\omega}\frac{\cosh k_0(h+z)}{\cosh k_0 h}\mathrm{e}^{\mathrm{i}k_0 x}$$

Radiation potential

$$\phi^{R} = -\sum_{l=0}^{\infty}\frac{\omega\chi^{A}D_n}{k_n C_n}\cosh k_n(h+z)\mathrm{e}^{\mathrm{i}k_n x}$$

Gate response

$$\chi^{A} = \frac{-\rho a A g D_0/(\epsilon\cosh k_0 h)}{-\omega^2 M + C - \mathrm{i}\omega^2\rho a\sum_{l=0}^{\infty}\frac{D_n^2}{k_n C_n}}$$

# Third order solution $O(\varepsilon^2)$ – zeroth harmonic

At this order we invoke the solvability condition applying Green's theorem

$$-\int_0^b \mathrm{d}y \int_0^X \mathcal{F}_{30}|_{z=0}\,\mathrm{d}x + \int_0^b \mathrm{d}y \int_{-h}^0 \mathcal{G}_{30}|_{x=0}\,\mathrm{d}z - \int_0^b \mathrm{d}y \int_{-h}^0 \phi_{10}\phi_{30_x}|_{x=X}\,\mathrm{d}z = 0$$

Where the ***forcing terms*** on the free surface and on the gate surface are given by

$$\mathcal{F}_{30} = -\omega^2\phi_{10_{t_1t_1}} + \frac{1}{g\epsilon}\left\{\frac{3\omega^4 f_{11}\omega(\chi\chi^*_{t_1} + \chi^*\chi_{t_1})}{g} + \omega f_{11}(\chi^*\phi_{21_{zz}} + \chi\phi^*_{21_{zz}}) - 2\omega|\chi|^2_{t_1}|\nabla f_{11}|^2 + \omega f_{11_z}[\chi^*(-\phi_{21} + \chi_{t_1}f_{11}) - \chi(\phi^*_{21} + \chi^*_{t_1}f_{11})]\right\},$$

$$\mathcal{G}_{30} = -\frac{r}{\epsilon}(\chi^*\phi_{21_{xx}} + \chi\phi^*_{21_{xx}}) + \omega\chi_{20_{t_1}}.$$

Solution of the integrals above gives $\phi_{10} = 0$ → ***No drift*** at the leading order

# Third order solution $O(\varepsilon^2)$ – First harmonic

At this order we invoke again the solvability condition applying Green's theorem

$$-\epsilon^2 \mathrm{i}\vartheta_{t_2} = \vartheta(c_A + \mathrm{i}c_B) + \vartheta^2\vartheta^*(c_N + \mathrm{i}c_R) + A\mathrm{e}^{(-\mathrm{i}c_\delta t_1/\epsilon)}(c_S + \mathrm{i}c_U) + \mathrm{i}\vartheta\nu c_L$$

If we assume a small **detuning** $2\Delta\omega$ with $\Delta\omega/\omega \sim \varepsilon^2$

$$-\mathrm{i}\overline{\vartheta}_t = \overline{\vartheta}(\Delta\omega + c_A + \mathrm{i}c_B) + \overline{\vartheta}^2\overline{\vartheta}^*(c_N + \mathrm{i}c_R) + A(c_S + \mathrm{i}c_U) + \mathrm{i}\overline{\vartheta}\nu c_L$$

where

$$\vartheta = \overline{\vartheta}\mathrm{e}^{-\mathrm{i}(\omega_2 t_2 + (t_1 c_\delta/\omega\epsilon))}$$

This is the evolution equation of the **Ginzburg-Landau type**. The coefficients $c_A$ and $c_B$ represent detuning and damping caused by the shape of the array, $c_N$ represents the shift of the eigenfrequency from the incident wave frequency, $c_R$ is the radiation damping due to wave radiation, $c_S$ and $c_U$ represent the energy influx by the incident waves while $c_L$ represents damping due to the PTO mechanism.

Usage of action-angle variables expressed by $\bar{\vartheta} = Re^{i\psi}$ yields

$$\left.\begin{aligned} R_t &= -R(c_L\nu + c_B) - R^3 c_R - A(c_U\cos\psi - c_S\sin\psi), \\ \psi_t &= \Delta\omega + R^2 c_N + \frac{A}{R}(c_S\cos\psi + c_U\sin\psi). \end{aligned}\right\}$$

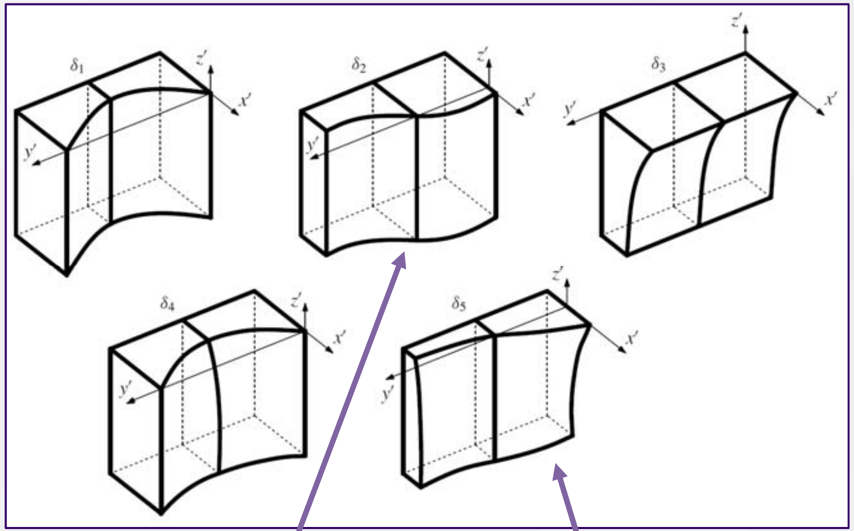
Non-trivial fixed points

$$-R(c_L\nu + c_B + R^2 c_R) + \sqrt{A^2(c_S^2 + c_U^2) - R^2(c_N R^2 + \Delta\omega + c_A)^2} = 0$$

One stable point or three roots, i.e. two stable points and an unstable saddle

# Results for uniform incident waves and *Q=2* gates

Depth: $h$=5 m; Gate width: $a$=5 m; Incident wave amplitude: $A$=0.1 m.

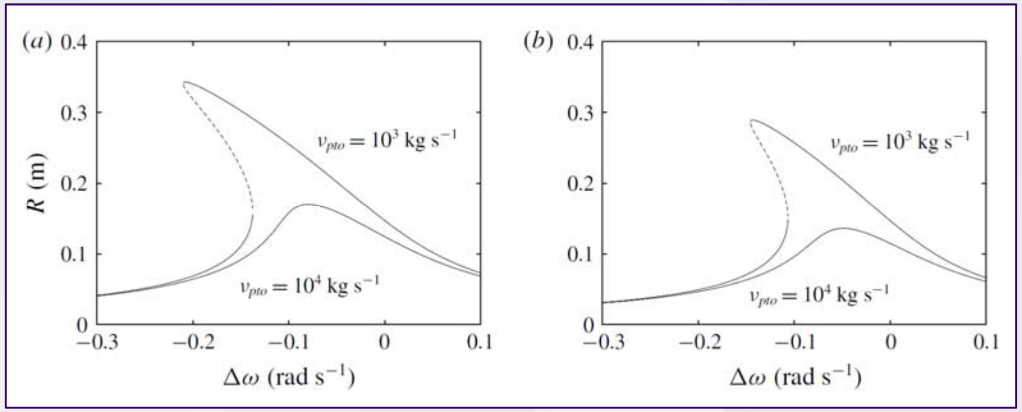



Nonlinear synchronous excitation is possible for $\delta_2$ and $\delta_5$

Gate shape functions

$$\left.\begin{aligned}\delta_1 &= -\frac{b}{10}\sin\frac{\pi y}{b}, \quad \delta_2 = \frac{b}{10}\cos\frac{\pi y}{b}, \quad \delta_3 = \frac{b\cosh 0.24(h+z)}{10\cosh 0.24h},\\ \delta_4 &= -\frac{b\sin\frac{\pi y}{b}\cosh 0.24(h+z)}{10\cosh 0.24h}, \quad \delta_5 = \frac{b\cos\frac{\pi y}{b}\cosh 0.24(h+z)}{10\cosh 0.24h}.\end{aligned}\right\}$$

Equilibrium branches ($\omega$=1.2 rad s$^{-1}$)



# Results for uniform incident waves and $Q=2$ gates

Generated power

$$P_{sync} = 2\nu_{pto}(\omega + \Delta\omega)^2 \sum_{q=1}^{Q} r_q^2 R^2$$

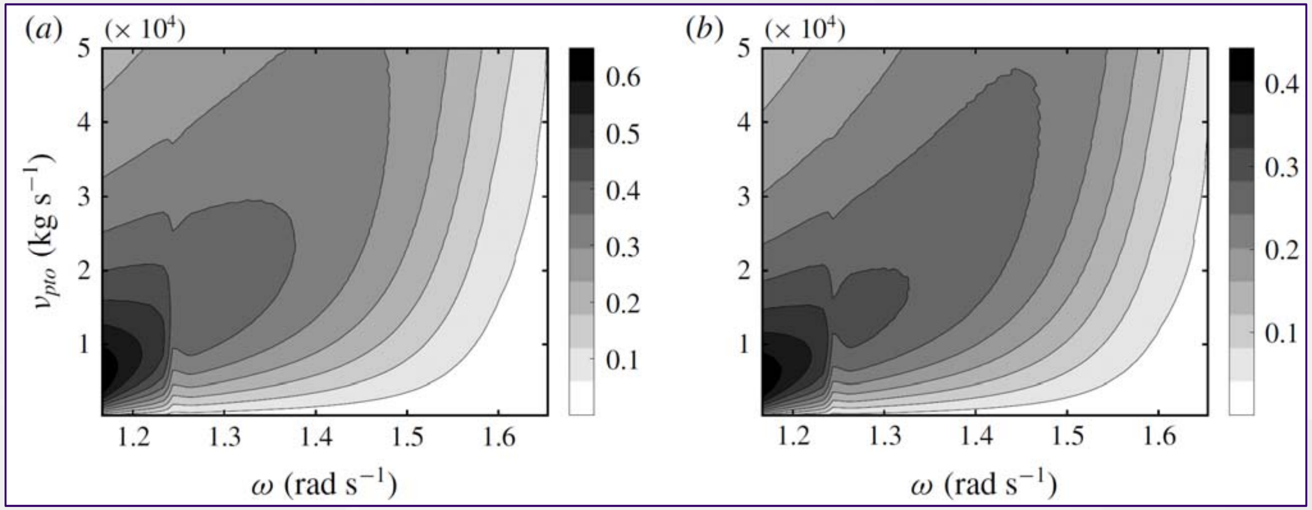
Capture factor

$$\mathcal{C}_{sync}^F = \frac{P_{sync}}{EC_g b}$$

Energy flux

$$EC_g = \frac{\rho g A^2(\omega + \Delta\omega)}{4k}\left(1 + \frac{2kh}{\sinh 2kh}\right)$$

Behaviour of the ***maximum value*** of the capture factor due to nonlinear synchronous resonance




In the presence of small-amplitude incident waves and trapped modes, a device designed to resonate synchronously can still achieve significant efficiency.

# Subharmonic resonance

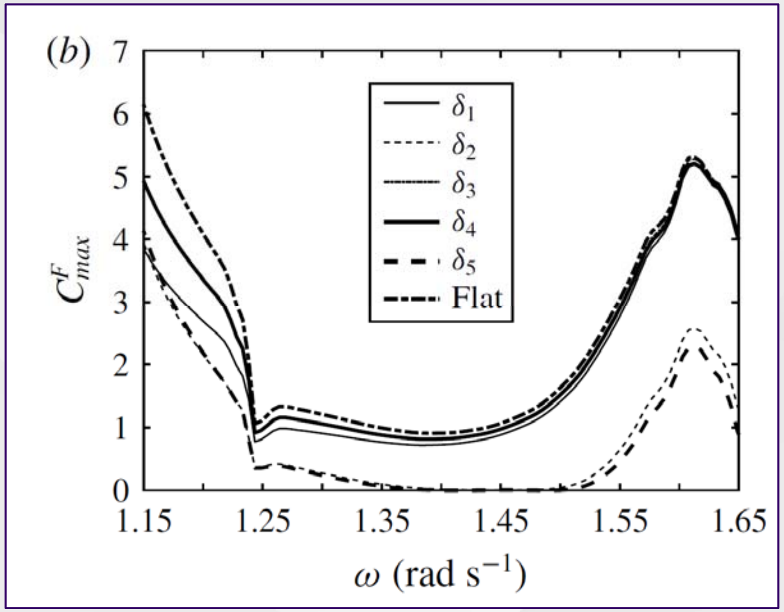
The evolution equation is similar to the synchronous case and admits both trivial and non-trivial fixed points

$$-i\overline{\vartheta}_t = \overline{\vartheta}(\Delta\omega + c_A + ic_B) + \overline{\vartheta}^2\overline{\vartheta}^*(c_N + ic_R) + A\overline{\vartheta}^*(c_F + ic_T) + i\overline{\vartheta}\nu c_L$$

By using $\overline{\vartheta} = i\sqrt{R}e^{i\psi}$

We obtain the unstable ($R^-$) and stable branches ($R^+$)

$$R^{\pm} = \frac{1}{c_N^2 + c_R^2}\left\{-c_R(\nu c_L + c_B) - c_N(\Delta\omega + c_A) \pm\sqrt{A^2(c_F^2 + c_T^2)(c_N^2 + c_R^2) - [c_N(\nu c_L + c_B) - c_R(\Delta\omega + c_A)]^2}\right\}$$




The flat configuration is the most efficient. This is because the coefficient $c_B$ generates hydrodynamic damping.

# Thank you for your attention

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## References

- MICHELE, S., RENZI, E. & SAMMARCO, P. 2019 Weakly nonlinear theory for a gate-type curved array in waves. *J. Fluid Mech.* 869, 238–263.
- MICHELE, S., SAMMARCO, P. & D'ERRICO, M. 2018 Weakly nonlinear theory for oscillating wave surge converters in a channel. *J. Fluid Mech.* 834, 55–91.