Weakly nonlinear theory of an array of curved wave energy converters

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Today you will see…

… how we can change the shape of an Oscillating Wave Surge Converter (OWSC) to maximize energy production.

The gate model is similar to that of Carrier (1970) except for a weak horizontal deviation of the gate wetted surface about the vertical plane.
State of the art

• The hydrodynamics of flaps has been extensively analysed in the past. Hereafter few works:

  • Plates

  • Venice gates
    • Li & Mei (2003), Mei et al. (1994), Adamo & Mei (2005).

  • OWSCs in a channel

  • OWSCs in open sea

  • Nonlinear theories
    • Sammarco et al. (1997), Vittori et al. (1996).

  • And so on…
Motivation of the study

• Most wave theories applied to OWSCs neglect \textit{nonlinear} hydrodynamic-related terms

• Recently, Michele \textit{et al.} (2018) showed that occurrence of \textit{subharmonic resonance} and \textit{mode competition} of trapped modes increases efficiency

Analytical solutions exist for simple geometries (rectangular, circular, elliptical, …)

• In this study we analyse the effects of more \textit{complex geometries} in \textit{nonlinear regimes}
Mathematical model

Gate surface equation

\[ x' - X'(y', t') - \delta'(y', z') = 0 \]

Non-dimensional quantities

\[
\begin{align*}
(x, y, z) &= (x', y', z')/\lambda', & \Phi &= \Phi'/A_T^2 \lambda', & \zeta &= \zeta'/A_T, & t &= t' \omega', \\
(a, b, h) &= (a', b', h')/\lambda, & X &= X'/A_T, & \delta &= \delta'/\delta_g, & G &= g'/(\omega^2 \lambda'),
\end{align*}
\]

Two small parameters

\[ \epsilon = A_T'/\lambda' \ll 1, \quad \mu = \delta_g'/\lambda' \ll 1, \quad \mu = O(\epsilon). \]

Governing equation

\[ \nabla^2 \Phi = 0 \]

Boundary conditions on the free surface

\[
\begin{align*}
- G\zeta &= \Phi_t + \epsilon \frac{1}{2} |\nabla \Phi|^2, & z &= \epsilon \zeta, \\
\Phi_n + G\Phi_z + \epsilon |\nabla \Phi|^2 + \epsilon^2 \frac{1}{2} \nabla \Phi \cdot \nabla |\nabla \Phi|^2 &= 0, & z &= \epsilon \zeta,
\end{align*}
\]

No-flux boundary conditions

\[
\begin{align*}
\Phi_z &= 0, & z &= -h, \\
\Phi_y &= 0, & y &= 0 \quad \text{and} \quad y = b.
\end{align*}
\]
Mathematical model

Kinematic condition on the array surface

\[ \Phi_x = X_t + \mu (\Phi_y \delta_y + \Phi_z \delta_z) \quad x = \epsilon X + \mu \delta \]

Equation of motion of the \( q \)th gate coupled with a linear damper

\[
\varepsilon M X_{q,t} + \varepsilon C G X_q + \varepsilon^3 v X_{q,t} = \int_{(q-1)a}^{qa} dy \left\{ \int_{-1}^{\varepsilon} dz \left( \epsilon \Phi_t + \epsilon^2 \frac{1}{2} |\nabla \Phi|^2 \right) + \int_{0}^{\varepsilon} G z \ dz \right\}
\]

\[
M = M' / (\rho' \lambda^3)
\]
Non-dimensional mass

\[
C = C' / (g' \rho' \lambda^2)
\]
Non-dimensional stiffness

\[
v = v' / (A_t^2 \omega' \rho' \lambda')
\]
Non-dimensional PTO coefficient
Multiple-scale analysis and three timing

Let us introduce the following expansion for the unknowns

\[
\Phi = \Phi_1(x, y, z, t, t_1, t_2) + \varepsilon \Phi_2(x, y, z, t, t_1, t_2) + \varepsilon^2 \Phi_3(x, y, z, t, t_1, t_2) + O(\varepsilon^3),
\]

\[
\zeta = \zeta_1(x, y, t, t_1, t_2) + \varepsilon \zeta_2(x, y, t, t_1, t_2) + \varepsilon^2 \zeta_3(x, y, t, t_1, t_2) + O(\varepsilon^3),
\]

\[
X_q = X_{q,1}(t, t_1, t_2) + \varepsilon X_{q,2}(t, t_1, t_2) + \varepsilon^2 X_{q,3}(t, t_1, t_2) + O(\varepsilon^3),
\]

\[
X = X_1(y, t, t_1, t_2) + \varepsilon X_2(y, t, t_1, t_2) + \varepsilon^2 X_3(y, t, t_1, t_2) + O(\varepsilon^3),
\]

Three-timing is necessary to avoid secularity at the second and third order

\[ t_1 = \varepsilon t \quad t_2 = \varepsilon^2 t \]

Higher order solutions imply higher harmonics. Return in physical variables and assume the following expansion

\[
\{\Phi_n, \zeta_n, X_{q,n}, X_n\} = \sum_{m=0}^{n} \{\phi_{nm}, \eta_{nm}, \chi_{q,nm}, \chi_{nm}\} e^{-in\omega t} + * \]
Multiple-scale analysis and three timing

The nonlinear set of governing equations and boundary conditions is decomposed in a sequence of linear boundary-value problems of order $n$ and harmonic $m$.

The forcing terms $F_{nm}$, $B_{nm}$, $G_{nm}$ and $D_{nm}$ are defined for each order.
Leading order solution $O(1)$

Zero-th harmonic unforced with homogeneous b.c.

First harmonic yields the trapped mode solution

$$\phi_{11} = i\chi \omega \sum_{q=1}^{Q} \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} b_{mq} D_n \frac{e^{-\alpha_{nm}x}}{C_n} \cos \frac{m\pi y}{b} \cosh k_n(h+z) \equiv i\chi f_{11}(x, y, z)$$

Displacement relation

$$\omega^2 = gk_0 \tanh k_0 h, \quad \omega^2 = -gk_n \tanh k_n h, \quad k_n = i\kappa_n, n = 1, \ldots, \infty.$$  

Real coefficients

$$b_{mq} = r_q \frac{2}{m\pi} \left[ \sin \frac{q\pi}{Q} - \sin \frac{(q-1)m\pi}{Q} \right],$$

$$\alpha_{nm} = \sqrt{\left( \frac{m\pi}{b} \right)^2 - k_n^2}, \quad C_n = \frac{1}{2} \left( h + \frac{g}{\omega^2} \sinh^2 k_n h \right), \quad D_n = \frac{\sinh k_n h}{k_n}.$$  

Solution of the equation of motion gives ($Q-1$) out-of-phase natural trapped modes and related eigenfrequencies.
Leading order solution $O(1)$

- Examples of $(Q-1)$ trapped modes for an array of $Q=5$ gates
- The number over each gate represents the normalized amplitude

- Even modes

- Odd modes
**Second order problem \( O(\varepsilon) \): zero-th harmonic**

Forcing terms on the free surface and on the gate surface yield a second order drift

\[
\eta_{20} = -\frac{i}{g\varepsilon} \left[ \phi_{100} \varepsilon \omega + |\chi|^2 (f_{11x}^2 + f_{11y}^2 - f_{11z}^2) \right],
\]

\[
\chi_{q,20} = \frac{\rho}{C\varepsilon} \int_{(q-1)\alpha}^{q\alpha} dy \left\{ -\frac{\omega^2 |\chi|^2}{g} f_{11} \right\}_{z=0} + \int_{-h}^{0} dz \left[ \phi_{100} \varepsilon \omega + |\chi|^2 (-f_{11x}^2 + f_{11y}^2 + f_{11z}^2) \right]_{z=0}
\]

- Bound wave
- No dependence on the gate shape \( \delta \)
- \( X_{20} \) does not affect power extraction

**Second order problem \( O(\varepsilon) \): first harmonic**

The gate shape 'forces' the first harmonic problem

**Solvability condition**

\[
\int \int \int_{\Omega} (\phi_{11} \nabla^2 \phi_{21} - \phi_{21} \nabla^2 \phi_{11}) \, d\Omega = \int \int_{\partial\Omega} \left( \phi_{11} \frac{\partial \phi_{21}}{\partial n} - \phi_{21} \frac{\partial \phi_{11}}{\partial n} \right) \, dS = 0
\]
Second order problem $O(\varepsilon)$: first harmonic – the gate shape

Evolution equation

\[ \chi_{t_1} - \frac{ic_\delta}{\omega \varepsilon} \chi = 0 \]

$\delta = 0$ (flat gate) implies $c_\delta = 0$
thus $\chi$ depends on the slow time scale $t_2$ only

Solution

\[ \chi(t_1, t_2) = \bar{\psi}(t_2)e^{-ic_\delta t_1/\omega \varepsilon} = \bar{\psi}(t_2)e^{-ic_\delta t} \]

\[ c_\delta = \frac{1}{c_f} \int_0^b dy \int_{-h}^0 dz \{ f_{11}(f_{11x} \delta - f_{11y} \delta_x - f_{11z} \delta_z) + r \omega \delta f_{11x} \} \]

\[ c_f = \int_0^b dy \int_0^{+\infty} \frac{2\omega f_{11}^2}{g} dx + \int_0^b dy \int_{-h}^0 2f_{11} r dz + \sum_{q=1}^Q \frac{2a \omega M r_q^2}{\rho} \]

$c_\delta$ represents a modulation of the modal amplitude growth
Response to synchronous incident waves

This is a diffraction – radiation problem forced by the incident wave field. The gates move at unison in phase.

Linearity allows the following decomposition

\[ \phi_{A}^{\pm} = \phi^{I} + \phi^{S} + \phi^{R} \]

Velocity potential incident waves of amplitude \( A \)

\[ \phi^{I} = -\frac{iAg \cosh k_{0}(h+z)}{2\epsilon \omega \cosh k_{0}h} e^{-ik_{0}x} \]

Scattered wave potential

\[ \phi^{S} = -\frac{iAg \cosh k_{0}(h+z)}{2\epsilon \omega \cosh k_{0}h} e^{ik_{0}x} \]

Radiation potential

\[ \phi^{R} = -\sum_{l=0}^{\infty} \frac{\omega \chi_{A} D_{n}}{k_{n}C_{n}} \cosh k_{n}(h+z) e^{ik_{n}x} \]

Gate response

\[ \chi_{A} = \frac{-\rho aAgD_{0}/(\epsilon \cosh k_{0}h)}{-\omega^{2}M + C - i\omega^{2}\rho a \sum_{l=0}^{\infty} \frac{D_{n}^{2}}{k_{n}C_{n}}} \]
Third order solution $O(\varepsilon^2)$ – zeroth harmonic

At this order we invoke the solvability condition applying Green’s theorem

$$- \int_0^b dy \int_0^b dx \mathcal{F}_{30} |_{z=0} dx + \int_0^b dy \int_0^0 dz - \int_0^b dy \int_{-h}^{0} \phi_{10} \phi_{30x} |_{x=X} dz = 0$$

Where the **forcing terms** on the free surface and on the gate surface are given by

$$\mathcal{F}_{30} = -\omega^2 \phi_{101_1} + \frac{1}{g\varepsilon} \left\{ \frac{3\omega f_{11} \omega (\chi \chi_{t_1}^* + \chi^* \chi_{t_1})}{g} + \omega f_{11} (\chi^* \phi_{21z} + \chi \phi_{21z}^*) ight. \\
- 2 \omega |\chi|_{t_1}^2 |\nabla f_{11}|^2 + \omega f_{11} [\chi^*(-\phi_{21} + \chi_{t_1} f_{11}) - \chi (\phi_{21}^* + \chi_{t_1} f_{11})] \right\},$$

$$\mathcal{G}_{30} = -\frac{r}{\varepsilon} (\chi^* \phi_{21x} + \chi \phi_{21x}^*) + \omega \chi_{201_1}.$$

Solution of the integrals above gives $\phi_{10} = 0$ **No drift** at the leading order
Third order solution $O(\varepsilon^2)$ – First harmonic

At this order we invoke again the solvability condition applying Green’s theorem

\[-\varepsilon^2 i \vartheta_t = \vartheta (c_A + ic_B) + \vartheta^2 \vartheta^* (c_N + ic_R) + A e^{(-i\omega t_1/\varepsilon)} (c_S + ic_U) + i \vartheta v c_L\]

If we assume a small detuning $2\Delta\omega$ with $\Delta\omega/\omega \sim \varepsilon^2$

\[-i \vartheta_t = \bar{\vartheta} (\Delta\omega + c_A + ic_B) + \bar{\vartheta}^2 \bar{\vartheta}^* (c_N + ic_R) + A (c_S + ic_U) + i \bar{\vartheta} v c_L\]

where

\[\vartheta = \bar{\vartheta} e^{-i(\omega t_2 + (t_1 c_R/\varepsilon))}\]

This is the evolution equation of the Ginzburg-Landau type. The coefficients $c_A$ and $c_B$ represent detuning and damping caused by the shape of the array, $c_N$ represents the shift of the eigenfrequency from the incident wave frequency, $c_R$ is the radiation damping due to wave radiation, $c_S$ and $c_U$ represent the energy influx by the incident waves while $c_L$ represents damping due to the PTO mechanism.

Usage of action-angle variables expressed by $\vartheta = Re^{i\psi}$ yields

Non-trivial fixed points

\[-R(c_L v + c_B + R^2 c_R) + \sqrt{A^2(c_S^2 + c_U^2) - R^2(c_N R^2 + \Delta\omega + c_A)^2} = 0\]

One stable point or three roots, i.e. two stable points and an unstable saddle
Results for uniform incident waves and $Q=2$ gates

Depth: $h=5$ m; Gate width: $a=5$ m; Incident wave amplitude: $A=0.1$ m.

Nonlinear synchronous excitation is possible for $\delta_2$ and $\delta_5$.

Gate shape functions

$$
\begin{align*}
\delta_1 &= \frac{b}{10} \sin \frac{\pi y}{b}, \\
\delta_2 &= \frac{b}{10} \cos \frac{\pi y}{b}, \\
\delta_3 &= \frac{b \cosh 0.24(h+z)}{10 \cosh 0.24h}, \\
\delta_4 &= -\frac{b \sin \frac{\pi y}{b} \cosh 0.24(h+z)}{10 \cosh 0.24h}, \\
\delta_5 &= \frac{b \cos \frac{\pi y}{b} \cosh 0.24(h+z)}{10 \cosh 0.24h}.
\end{align*}
$$

Equilibrium branches ($\omega=1.2$ rad s$^{-1}$)
### Results for uniform incident waves and $Q=2$ gates

<table>
<thead>
<tr>
<th>Generated power</th>
<th>Capture factor</th>
<th>Energy flux</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_{\text{sync}} = 2v_{\text{pre}}(\omega + \Delta \omega)^2 \sum_{q=1}^{Q} r_q^2 R^2$</td>
<td>$C_{\text{sync}}^{F} = \frac{P_{\text{sync}}}{EC_g b}$</td>
<td>$EC_g = \frac{\rho g A^2 (\omega + \Delta \omega)}{4k} \left(1 + \frac{2kh}{\sinh 2kh}\right)$</td>
</tr>
</tbody>
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Behaviour of the *maximum value* of the capture factor due to nonlinear synchronous resonance.

In the presence of small-amplitude incident waves and trapped modes, a device designed to resonate synchronously can still achieve significant efficiency.
Subharmonic resonance

The evolution equation is similar to the synchronous case and admits both trivial and non-trivial fixed points

\[-i\overline{\vartheta}_t = \overline{\vartheta}(\Delta\omega + c_A + ic_B) + \overline{\vartheta}^2\overline{\vartheta}^* (c_N + ic_R) + A\overline{\vartheta}^* (c_F + ic_T) + i\overline{\vartheta} \nu c_L\]

By using \(\overline{\vartheta} = i\sqrt{R}e^{i\psi}\)

We obtain the unstable \((R^-)\) and stable branches \((R^+)\)

\[R^\pm = \frac{1}{c^2_N + c^2_R} \left\{ -c_R(\nu c_L + c_B) - c_N(\Delta\omega + c_A) \right. \]

\[\left. \pm \sqrt{A^2(c^2_F + c^2_T)(c^2_N + c^2_R) - [c_N(\nu c_L + c_B) - c_R(\Delta\omega + c_A)]^2} \right\}\]

The flat configuration is the most efficient. This is because the coefficient \(c_B\) generates hydrodynamic damping.
Thank you for your attention

References
