Wave generation techniques using the Boussinesq-Abbott & Green-Naghdi equations

Lisl Weynans[1] (joint work with D. Lannes)
Maria Kazolea [2]

IMB [1]
INRIA Boredeaux Sud-Ouest[2]
Cardamom team

June 18 2019

Dimensionless Nonlinear Boussinesq Equations

- Dimensionless Boussinesq system

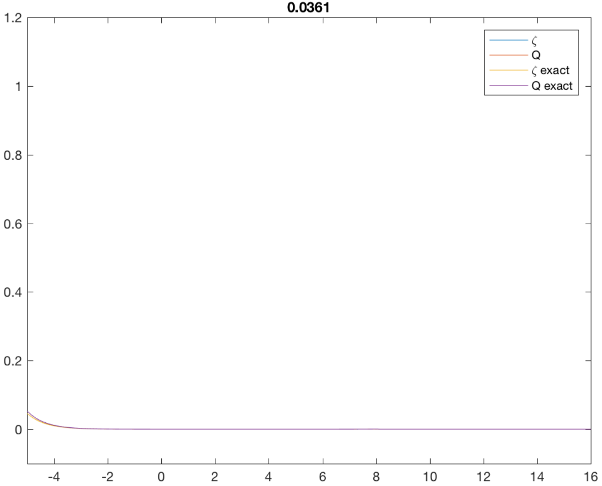
$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{\mu}{3}\partial_x^2)\partial_t q + \partial_x\Big(\frac{1}{2\varepsilon}h^2 + \varepsilon\frac{1}{h}q^2\Big) = 0, \end{cases}$$

- Coupling surface elevation ζ above rest state to horizontal discharge q
- ε and μ respectively called *nonlinearity* and *shallowness* parameters
- Initial conditions

$$(\zeta, q)(t = 0, x) = (\zeta^0, q^0)(x)$$

$\Rightarrow$ How to impose the generating boundary condition $\zeta(t, x = 0) = f(t)$?

Propagation of solitary wave



Propagation of the soliton, $N_x = 200$

Generating boundary condition ?

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{\mu}{3}\partial_x^2)\partial_t q + \partial_x\Big(\frac{1}{2\varepsilon}h^2 + \varepsilon\frac{1}{h}q^2\Big) = 0, \end{cases}$$

with initial and boundary conditions

$$(\zeta, q)(t = 0, x) = (\zeta^0, q^0)(x), \qquad \zeta(t, x = 0) = f(t).$$

- For periodic boundary conditions : inversion of operator $(1 - \frac{\mu}{3}\partial_x^2)$ ok
- For a generating boundary condition : not possible to use Riemann invariants as in Nonlinear Shallow Water case
- Inversion of $(1 - \frac{\mu}{3}\partial_x^2)$ on the half-line $(0, \infty)$: need of a boundary condition on $\partial_t q$!

Generating boundary condition

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \frac{\mu}{3}\partial_x^2)\partial_t q + \partial_x\Big(\frac{1}{2\varepsilon}h^2 + \varepsilon \frac{1}{h} q^2\Big) = 0, \end{cases}$$

with initial and boundary conditions

$$(\zeta, q)(t = 0, x) = (\zeta^0, q^0)(x), \qquad \zeta(t, x = 0) = f(t).$$

Strategy :

- use the inverse of operator $(1 - \frac{\mu}{3}\partial_x^2)$ with homogeneous Dirichlet BC
- construct the dispersive boundary layer accounting for the non-zero boundary value of q

Reformulation of the Boussinesq equations as a system

- with a nonlocal flux
- and a source term accounting for the dispersive boundary layer

Nonlinear Boussinesq Equations

Definition

We denote by R_0 and R_1 the inverse of the operator $(1 - \frac{\mu}{3}\partial_x^2)$ with homogeneous Dirichlet and Neumann boundary conditions :

$$R_0 : \begin{array}{ccc} L^2(\mathbb{R}_+) & \to & H^2(\mathbb{R}_+) \\ f & \mapsto & u, \end{array} \quad \text{where} \quad \begin{cases} (1 - \frac{\mu}{3}\partial_x^2)u = f \\ u(0) = 0 \end{cases}$$

and

$$R_1 : \begin{array}{ccc} L^2(\mathbb{R}_+) & \to & H^2(\mathbb{R}_+) \\ f & \mapsto & v, \end{array} \quad \text{where} \quad \begin{cases} (1 - \frac{\mu}{3}\partial_x^2)v = f \\ \partial_x v(0) = 0 \end{cases} .$$

Lemma

For all $f \in L^2(\mathbb{R}_+)$,

$$R_0 \partial_x f = \partial_x R_1 f.$$

Notation : $\underline{R}_1 f = (R_1 f)_{|x=0}$ and $q(t) = q(t, x = 0)$

Boussinesq Equations : dispersive boundary layer

$$\begin{cases}\partial_t\zeta + \partial_x q = 0,\\ (1-\frac{\mu}{3}\partial_x^2)\partial_t q + \partial_x\Big(\mathfrak{f}(\zeta,q)\Big) = 0, \quad \text{with } \mathfrak{f}(\zeta,q) = \frac{1}{2\varepsilon}h^2 + \varepsilon\frac{1}{h}q^2\end{cases}$$

- The ODE

$$Y - \frac{\mu}{3}Y'' = f, \qquad Y(0) = Y_0$$

admits a unique solution given by

$$Y(x) = R_0 f + Y_0 \exp\Big(-\frac{x}{\delta}\Big) \quad \text{with} \quad \delta = \sqrt{\frac{\mu}{3}},$$

- Thus the second equation of Boussinesq system can be written equivalently

$$\partial_t q = \quad -R_0\partial_x\Big(\mathfrak{f}(\zeta,q)\Big) \quad + \quad \underbrace{\underline{\dot{q}}\exp\Big(-\frac{x}{\delta}\Big)}_{\text{dispersive boundary layer}} \quad .$$

$\Rightarrow$ Need to compute $\underline{\dot{q}}$

How to compute $\dot{\underline{q}}$?

$$\partial_t q + \partial_x R_1\big(\mathfrak{f}(\zeta, q)\big) = \dot{\underline{q}} \exp\Big(-\frac{x}{\delta}\Big).$$

- Differenciating with respect to x

$$\partial_t \partial_x q + \partial_x^2 R_1\big(\mathfrak{f}(\zeta, q)\big) = -\frac{1}{\delta}\dot{\underline{q}} \exp\Big(-\frac{x}{\delta}\Big).$$

- Substitution $\partial_t \partial_x q = -\partial_t^2 \zeta$

$$-\partial_t^2 \zeta + \partial_x^2 R_1\big(\mathfrak{f}(\zeta, q)\big) = -\frac{1}{\delta}\dot{\underline{q}} \exp\Big(-\frac{x}{\delta}\Big).$$

- Using $(1 - \frac{\mu}{3}\partial_x^2)R_1 = \mathrm{Id}$

$$\partial_t^2 \zeta = \frac{1}{\delta^2}(R_1 - \mathrm{Id})\big(\mathfrak{f}(\zeta, q)\big) + \frac{1}{\delta}\dot{\underline{q}} \exp\Big(-\frac{x}{\delta}\Big).$$

- At $x = 0$, formula for $\dot{\underline{q}}$:

$$\ddot{\underline{\zeta}} = \frac{1}{\delta^2}\Big[\underline{R_1}\big(\mathfrak{f}(\zeta, q)\big) - \mathfrak{f}(\underline{\zeta}, \underline{q})\Big] + \frac{1}{\delta}\dot{\underline{q}}$$

Equivalent form of Boussinesq Equations

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \partial_x R_1 \mathfrak{f}(\zeta, q) = \underline{Q}(\underline{q}, f, \ddot{f}, \zeta, q) \exp\left(-\frac{x}{\delta}\right), \\ \underline{\dot{q}} = \underline{Q}(\underline{q}, f, \ddot{f}, \zeta, q), \end{cases}$$

with

$$\mathfrak{f}(\zeta, q) = \frac{1}{2\varepsilon}(h^2 - 1) + \varepsilon \frac{1}{h} q^2 \quad (= \text{flux for NLSW momentum })$$

$$\underline{Q}(\underline{q}, f, \ddot{f}, \zeta, q) = \frac{\varepsilon}{\delta} \frac{\underline{q}^2}{1 + \varepsilon f} + \delta \ddot{f} + \frac{1}{\delta}(1 + \frac{\varepsilon}{2} f) f - \frac{1}{\delta} \underline{R}_1 \mathfrak{f}(\zeta, q),$$

+ initial data and and boundary condition $\zeta(t, x = 0) = f(t)$.

Discretization

- Lax-Friedrichs scheme for U_i^n, explicit Euler scheme for the ODE on $\underline{q}$,
- Non-local flux $\mathfrak{f}_\mu(U) = R_1 \mathfrak{f}(U)$ computed with second-order centered finite-differences

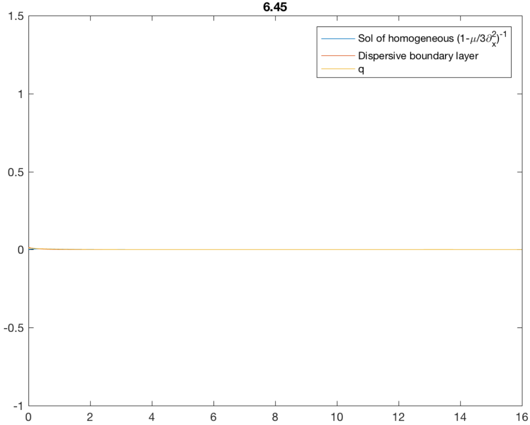
$$\begin{cases} \dfrac{U_i^{n+1} - U_i^n}{\delta_t} + \dfrac{1}{\delta_x}\left(\mathfrak{F}^n_{\mu,i+1/2} - \mathfrak{F}^n_{\mu,i-1/2}\right) = S_i^n, \quad i \geq 1, \quad n \geq 1, \\ \dfrac{\underline{q}^{n+1} - \underline{q}^n}{\delta_t} = \underline{Q}(\underline{q}^n, f^n, \ddot{f}^n, \zeta^n, q^n) \qquad n \geq 1, \end{cases}$$

where $U^n = (\zeta_i^n, q_i^n)^T_{i\geq 1}$, $\mathfrak{F}_\mu(U) = \left(q, \mathfrak{f}_\mu(U)\right)^T$ and

$$S_i^n = \begin{pmatrix} 0 \\ \underline{Q}(\underline{q}^n, f^n, \ddot{f}^n, \zeta^n, q^n)\exp(-\frac{x_i}{\delta}) \end{pmatrix}$$

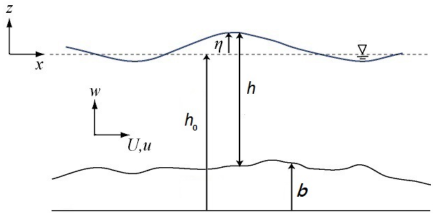
- Computational cost similar to the case of periodic conditions

Propagation of solitary wave



Decomposition of $\partial_t q$ into $\partial_t q = -\partial_x R_1\Big(\mathfrak{f}(\zeta, q)\Big) + \underline{\dot{q}} \exp\Big(-\frac{x}{\delta}\Big)$ (dispersive boundary layer and solution of homogenous problem)

The physical problem



η : **free surface elevation** ;
h_0 : **steel water level** ;
$b(x)$: **bottom's topography variation** ;
$h(x,t) = h_0 + \eta(x,t) - b(x)$: **total water depth** ;
$u(x,t)$: **flow velocity** ;

Mathematical model : The SW & GN equations

One dimensional form :

$$
\begin{aligned}
h_t + (hu)_x &= 0\\
(I+\alpha\mathcal{T})\left[(hu)_t + (hu^2)_x + g\frac{\alpha-1}{\alpha}h\eta_x\right] + \frac{g}{\alpha}h\eta_x + h\tilde{\mathcal{Q}}(u) &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{T}(\cdot) &= -\frac{1}{3}h^2(\cdot)_{xx} - \frac{1}{3}hh_x(\cdot)_x + \frac{1}{3}\left[h_x^2 + hh_{xx}\right](\cdot) + \left[b_x h_x + \frac{1}{2}hb_{xx} + b_x^2\right](\cdot)\\
\tilde{\mathcal{Q}}(\cdot) &= 2hh_x(\cdot)_x^2 + \frac{4}{3}h^2(\cdot)_x(\cdot)_{xx} + b_x h(\cdot)_x^2 + b_{xx}h(\cdot)(\cdot)_x\\
&+ \left[b_{xx}h_x + \frac{1}{2}hb_{xxx} + b_x b_{xx}\right](\cdot)^2
\end{aligned}
$$

Mathematical model : The SW & GN equations

One dimensional form :

$$
\begin{aligned}
h_t + (hu)_x &= 0\\
(I+\alpha\mathcal{T})\left[(hu)_t + (hu^2)_x + g\frac{\alpha-1}{\alpha}h\eta_x\right] + \frac{g}{\alpha}h\eta_x + h\tilde{\mathcal{Q}}(u) &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{T}(\cdot) &= -\frac{1}{3}h^2(\cdot)_{xx} - \frac{1}{3}hh_x(\cdot)_x + \frac{1}{3}\left[h_x^2 + hh_{xx}\right](\cdot) + \left[b_x h_x + \frac{1}{2}hb_{xx} + b_x^2\right](\cdot)\\
\tilde{\mathcal{Q}}(\cdot) &= 2hh_x(\cdot)_x^2 + \frac{4}{3}h^2(\cdot)_x(\cdot)_{xx} + b_x h(\cdot)_x^2 + b_{xx}h(\cdot)(\cdot)_x\\
&+ \left[b_{xx}h_x + \frac{1}{2}hb_{xxx} + b_x b_{xx}\right](\cdot)^2
\end{aligned}
$$

Dispersion relation

$$
\omega^2 = gh_0k^2\frac{1+\frac{\alpha-1}{3}k^2h_0^2}{1+\frac{\alpha}{3}k^2h_0^2}.
$$

where $\alpha = 1.159$

Mathematical model : Elliptic-hyperbolic decoupling

Re-write the system as :

$$
\begin{aligned}
h_t + (hu)_x &= 0 \\
(I + \alpha\mathcal{T})\Big[(hu)_t + (hu^2)_x + gh\eta_x\Big] - \mathcal{T}\,(gh\eta_x) + h\tilde{\mathcal{Q}}(u) &= 0
\end{aligned}
$$

Mathematical model : Elliptic-hyperbolic decoupling

Re-write the system as :

$$
\begin{aligned}
h_t + (hu)_x &= 0 \\
(I + \alpha\mathcal{T})\left[(hu)_t + (hu^2)_x + gh\eta_x\right] - \mathcal{T}(gh\eta_x) + h\tilde{Q}(u) &= 0
\end{aligned}
$$

$$
\begin{aligned}
(I + \alpha\mathcal{T})\,\phi &= \mathcal{W} - \mathcal{R} \\
h_t + (hu)_x &= 0 \\
(hu)_t + (hu^2)_x + gh\eta_x &= \phi
\end{aligned}
$$

where $\mathcal{W} = g\mathcal{T}(h\eta_x)$ and $\mathcal{R} = h\tilde{Q}(u)$.

Mathematical model : Elliptic-hyperbolic decoupling

Re-write the system as :

$$\begin{aligned} h_t + (hu)_x &= 0 \\ (I + \alpha\mathcal{T})\left[(hu)_t + (hu^2)_x + gh\eta_x\right] - \mathcal{T}(gh\eta_x) + h\tilde{\mathcal{Q}}(u) &= 0 \end{aligned}$$

$$\begin{aligned} (I + \alpha\mathcal{T})\,\phi &= \mathcal{W} - \mathcal{R} \\ h_t + (hu)_x &= 0 \\ (hu)_t + (hu^2)_x + gh\eta_x &= \phi \end{aligned}$$

where $\mathcal{W} = g\mathcal{T}(h\eta_x)$ and $\mathcal{R} = h\tilde{\mathcal{Q}}(u)$.

Two step solution

1. An elliptic step solving for the non-hydrostatic term ϕ ;
2. An hyperbolic step evolving the flow variables .

Structure of the code :

Go to : https ://mskazolea.wixsite.com/personal

files : main.m, flux.m, get_PHI.m, limiter.m, muscl.m, primitive.m, boundary.m wave_generation.m

Structure of the code :

Go to : https ://mskazolea.wixsite.com/personal

files : main.m, flux.m, get_PHI.m, limiter.m, muscl.m, primitive.m, boundary.m wave_generation.m

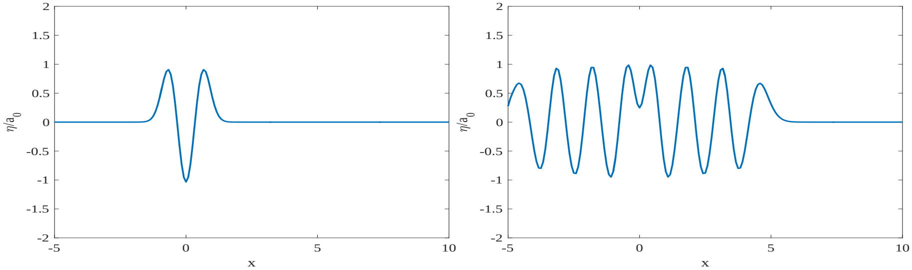
main.m :

- Input values provided by the users
- Intialization of the solution
- RK3 loop in time
  - For each RK step $p = 1, 2, 3$

$$\begin{cases} \text{Find the fluxes} \rightarrow \textit{flux.m}\text{(Third order FV scheme)} \\ \text{Update the value of } h^p \\ \text{Find the non Hydrostatic terms (if GN are solved)} \rightarrow \textit{get\_PHI.m}(C^0\text{Galerkin} \\ \text{Update the value of } (hu)^p \\ \text{Sponge layers} \rightarrow \textit{sponge.m} \end{cases}$$

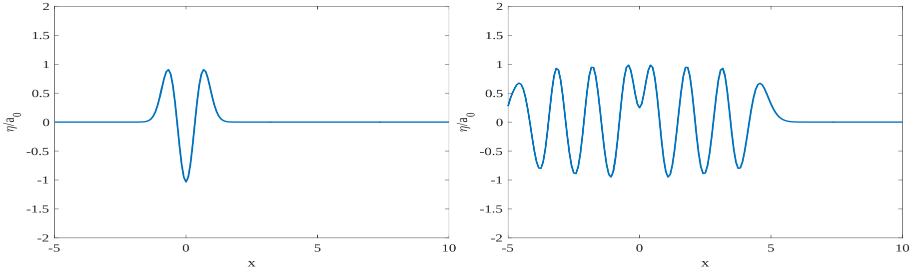
  - Find new dt

Wave generation : internal wave generation



- Common technique for BT models. (Wei &Kirby 1999)
- A source term added to the governing equations
  1. In a form of a mass source
  2. An applied pressure forcing the momentum equations

Wave generation : internal wave generation



- Common technique for BT models. (Wei &Kirby 1999)
- A source term added to the governing equations
  1. In a form of a mass source
  2. An applied pressure forcing the momentum equations
- Derivation from the linearized Boussinesq.
- Different tuning for different BT model.

Wave generation : internal wave generation

$$h_t + (hu)_x = f(x, t)$$

Wave generation : internal wave generation

$$h_t + (hu)_x = f(x,t)$$

where

$$f(x,t) = Dsin(-\omega t)exp(Bs(x - x0)^2)$$

for a given wave frequency ω wave direction θ, wave amplitude η_0 water depth h and source width parameter Bs the corresponding source amplitude D can be determined by

$$D = \frac{2\eta_0(\omega^2 - \alpha_1 g k^4 h^3)cos\theta}{\omega I_1 k\left[1 - \alpha(kh)^2\right]}$$

Absorbing boundaries : sponge layer approach

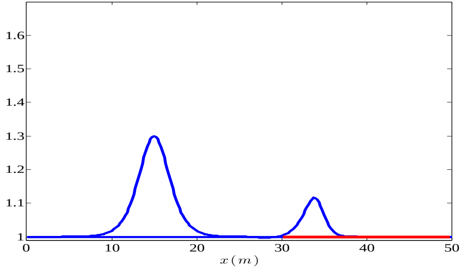
- They should dissipate the energy of incoming waves perfectly, in order to eliminate unphysical reflections.

Absorbing boundaries : sponge layer approach

- They should dissipate the energy of incoming waves perfectly, in order to eliminate unphysical reflections.
- A sponge layer is defined in-front of the physical boundary. First proposed by Larsen and Dancy 1983

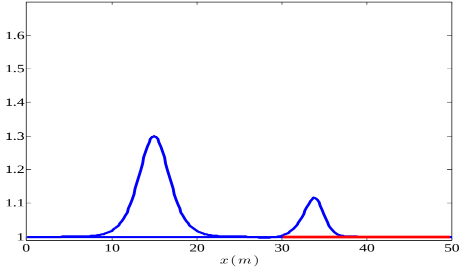
Absorbing boundaries : sponge layer approach

- They should dissipate the energy of incoming waves perfectly, in order to eliminate unphysical reflections.
- A sponge layer is defined in-front of the physical boundary. First proposed by Larsen and Dancy 1983



Absorbing boundaries : sponge layer approach

- They should dissipate the energy of incoming waves perfectly, in order to eliminate unphysical reflections.
- A sponge layer is defined in-front of the physical boundary. First proposed by Larsen and Dancy 1983



- On it the surface elevation and/or the fluxes are damped multiplying their values by a coefficient $\mu(x)$ after the computation of q^{n+1}
- Choice of $\mu(x)$?

Absorbing boundaries : sponge layer approach (cont)

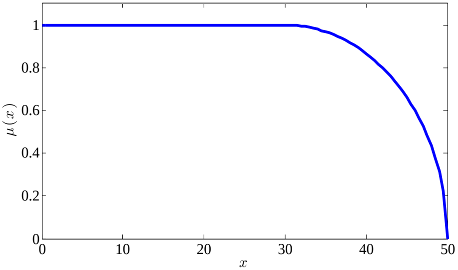
- Tonelli and Petti (2009) :

$$\mu(x) = \begin{cases} 0.5 + 0.5 * \cos\left(\pi \frac{L_s}{(L_s - D(x))}\right), & D \le L_s \\ 1, D \ge L_s \end{cases}$$

Absorbing boundaries : sponge layer approach (cont)

- Wu (2004) :

$$\mu(x) = \sqrt{1 - \left(\frac{1 - D(x)}{L_s}\right)^2}$$

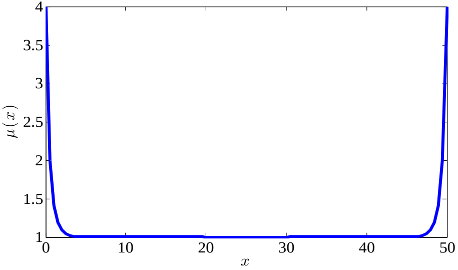


Ls-sponge layer width, D(x)=normal distance between the cell center with coordinates x and the absorbing boundary.

Absorbing boundaries : sponge layer approach (cont)

- Larsen and Dancy (1983)

$$\mu(x) = \begin{cases} \exp\left((2^{d/\Delta d} - 2^{-L_s/\Delta d}) \ln a\right), & 0 \leq d \leq L_s \\ 1, & d_s < d \end{cases}$$



d- distance between the point on the sponge layer and the boundary, Δd -dimension of the elements, a-parameter

Absorbing boundaries using extra damping terms

- Artificial damping terms F_b are added to the right hand side of the equation(s).
- General notation (Israeli and Orzag) :

$$F_b = -\omega_1(x)u + \omega_2(x)u_{xx} + \omega_3(x)\sqrt{\frac{g}{h}}\eta \tag{1}$$

where $\omega_i = c_i \omega f(x)$

$$f(x) = \frac{exp\left((x - x_s)/L_s\right)^2 - 1}{exp(1) - 1}$$

Absorbing boundaries using extra damping terms (cont)

- Different works use different coefficients e.g.
  - Kirby et al. use (1) mainly with $c_1 = 10, c_2 = c_3 = 0$; adding the terms only to the momentum equation.
  - Klonaris et al. add (1) only in the momentum equation with $c_3 = 0$
  - Filippini et al $\omega_1 = \omega_3 = 0$ and $\omega_2 = 0.1$ with

$$f(x) = \frac{exp\left((x - x_s)/L_s\right)^c - 1}{exp(1) - 1}$$

adding the terms to all the equations. c-parameter.

Absorbing boundaries using extra damping terms (cont)

- Different works use different coefficients e.g.
  - Kirby et al. use (1) mainly with $c_1 = 10, c_2 = c_3 = 0$; adding the terms only to the momentum equation.
  - Klonaris et al. add (1) only in the momentum equation with $c_3 = 0$
  - Filippini et al $\omega_1 = \omega_3 = 0$ and $\omega_2 = 0.1$ with

$$f(x) = \frac{exp\left((x - x_s)/L_s\right)^c - 1}{exp(1) - 1}$$

  adding the terms to all the equations. c-parameter.

- Attention ! on how these terms affect the stability of the scheme

References :

1. Israeli and Orzag, 1981. Approximation of radiation boundary conditions, J. Comp. Phys,41,15-135.
2. Kirby,J., Wei, G., Chen, Q., Kennedy, A. and Dalrymple, A. Funwave 1.0, 1998. Fully non-linear wave Model, Documentation and User's Manual.
3. A.G.Filippini, M. Kazolea and M. Ricchiuto, A flexible and genuinely nonlinear approach for nonlinear wave propagation , breaking and run-up, J.C.P., 130, 381-417,2016
4. Tonelli, M and Petti, M, 2009. Hybrid finite volume-finite difference scheme for 2HD improved Boussiesq equarions, Coastal Eng.,56.
5. Larsen, J. and Dancy, 1983.H. Open boundaries in short wave simulation- a new approach. Coastal Eng, 7, 285-297.
6. Wu, T.-R., 2004. A numerical study of three dimensional breaking waves and turbulence effects, Ph.D. thesis. Cornell University.