Wave-Structure interaction in shallow water

David Lannes
IMB, Université de Bordeaux et CNRS

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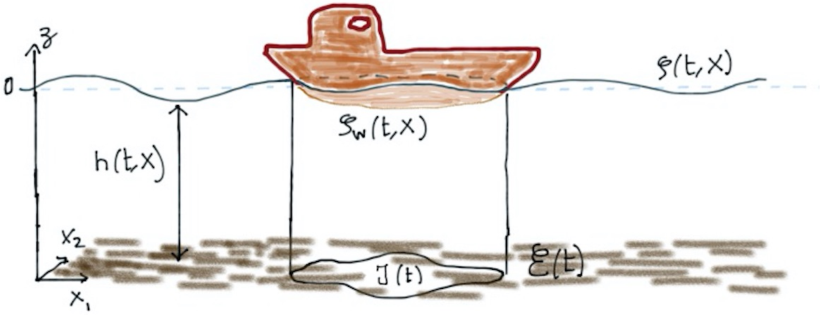
Outline

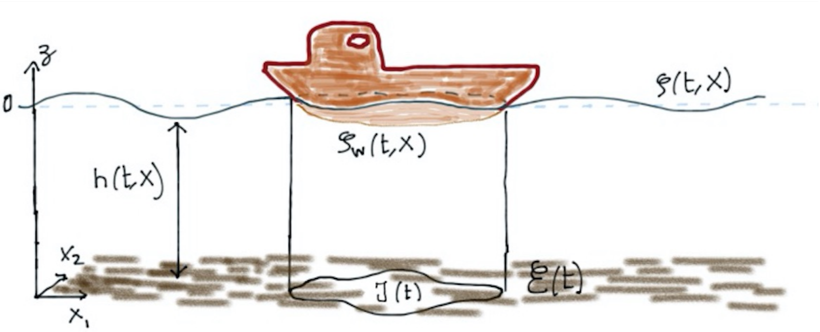
- General approach

General approach



Floating device: ship or wave energy convertor

General approach



Floating device: ship or wave energy convertor

Notation

If f is defined on $\mathbb{R}^d$, we write

$$f_{\rm i} = f_{|_{\mathcal{I}}}$$

In the fluid domain Ω_t

$$\begin{aligned} \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_{X,z} \mathbf{U} &= -\frac{1}{\rho} \nabla_{X,z} P - g \mathbf{e}_z \\ \mathrm{div}\ \mathbf{U} &= 0, \\ \mathrm{curl}\ \mathbf{U} &= 0 \end{aligned}$$

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At the surface

$$\boxed{\forall X \in \mathcal{E}(t),} \qquad \underline{P}(t,X) = P_{\rm atm},$$

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$$\forall X \in \mathbb{R}^d, \qquad \partial_t \zeta - \underline{U} \cdot N = 0 \qquad ,$$

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At the bottom

$$U_b \cdot N_b = 0.$$

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At the bottom

$$U_b \cdot N_b = 0.$$

Constraint in the interior domain

The surface of the fluid coincides with the wetted portion of the body

$$\zeta_{\mathrm{i}} = \zeta_{\mathrm{w}}$$

Coupling(s)

Interior/Exterior coupling on $\Gamma(t) := \partial\mathcal{I}(t) = \partial\mathcal{E}(t)$

Continuity of the surface elevation and of the surface pressure

$$\zeta(t,\cdot) = \zeta_{\rm i}(t,\cdot) \quad \text{and} \quad \underline{P}(t,\cdot) = \underline{P}_{\rm i}(t,\cdot) \quad \text{on} \quad \Gamma(t)$$

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Fluid/Solid coupling

Newton's equations:

$$\begin{cases} m\dot{U}_G & = -mg\mathbf{ez} + \int_{I(t)} (P_{\rm i} - P_{\rm atm}) N_{\rm w}, \\ \frac{d}{dt}(\mathcal{I}\omega) & = \int_{I(t)} (P_{\rm i} - P_{\rm atm}) \mathbf{r}_G \times N_{\rm w}. \end{cases}$$

Summary of the general approach

1. Euler equations with
   - Free surface, constrained pressure in the exterior domain
   - Constrained surface, free pressure in the interior domain
2. Coupling conditions at the contact line
3. Fluid/solid coupling via Newton's equations

Summary of the general approach

1. Euler equations with
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Remark

The same approach can be used with (simpler) asymptotic models:

1. *$1D$ shallow water models*
   $\rightsquigarrow$ *Free boundary problem for the contact line*
2. *$1D$ shallow water models and vertical walls*
   $\rightsquigarrow$ *Coupling conditions at the contact line*
3. *$1D$ Boussinesq models*
   $\rightsquigarrow$ *Dispersive boundary layer*

The one dimensional shallow water equations

In the exterior domain

$$\begin{cases} \partial_t H + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\frac{1}{H} Q^2 + \frac{1}{2} g H^2\right) = \frac{1}{\rho} H \partial_x \underline{P}_{\mathrm{atm}} = 0 \end{cases}$$

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In the interior domain $\mathcal{I} = (x_-(t), x_+(t))$

$$\begin{cases} \partial_x Q_{\rm i} = -\partial_t H_{\rm i}, \\ \partial_t Q_{\rm i} + \partial_x(\frac{1}{H_{\rm i}}Q_{\rm i}^2 + \frac{1}{2}gH_{\rm i}^2) = -\frac{1}{\rho}H_{\rm i}\partial_x \underline{P}_{\rm i}. \end{cases}$$

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Coupling conditions at $x = x_\pm(t)$

$$H(t,\cdot) = H_{\rm i}(t,\cdot), \qquad Q(t,\cdot) = Q_{\rm i}(t,\cdot), \qquad \text{and} \qquad \underline{P}_{\rm atm}(t,\cdot) = \underline{P}_{\rm i}(t,\cdot).$$

The one dimensional shallow water equations

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Coupling with the solid equations: the case of a fixed solid

$$\partial_t H_{\rm i} = 0 \quad \rightsquigarrow \quad Q_{\rm i}(t,x) = q_{\rm i}(t)$$

Reduction of the problem

- Interior equations

$$\begin{cases} Q(t,x) = q_{\mathrm{i}}, \\ \partial_t Q + \partial_x\left(\frac{1}{H}Q^2 + \frac{1}{2}gH^2\right) = -\frac{1}{\rho}H\partial_x \underline{P}_{\mathrm{i}} \end{cases}$$

with $\underline{P}_{\mathrm{i}}(t, x_{\pm}(t)) = P_{\mathrm{atm}}$

$\rightsquigarrow$ Solvability condition for $\underline{P}_{\mathrm{i}}$ $\left(\int_{x_-}^{x_+} \partial_x \underline{P}_{\mathrm{i}} = 0\right)$

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$$\partial_t q_{\rm i} = F(q_{\rm i}, x_+(t), x_-(t))$$

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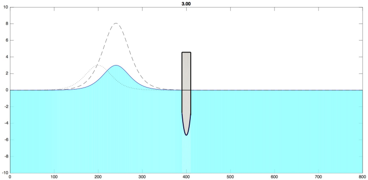
$$H(t, x_{\pm}(t)) = H_{\rm i}(t, x_{\pm}(t)) \qquad \text{(free boundary equation)}$$

$1D$ shallow water equations and vertical walls

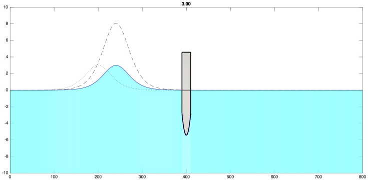


Fixed object with vertical sidewalls at $x_{\pm} = \pm R$

Equations in the exterior and interior domains

Unchanged: standard NSW equations (with pressure source term in $\mathcal{I}$)

$1D$ shallow water equations and vertical walls



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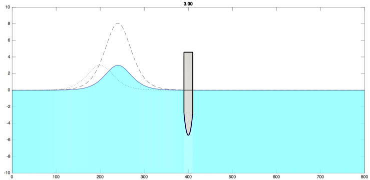

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Coupling with the solid equations: the case of a fixed solid

Unchanged

Non Vertical vs Vertical wals

Non vertical walls

- Continuity of Q $\rightsquigarrow$ Boundary condition for the exterior equations
- Continuity of H $\rightsquigarrow$ Evolution equation for $x_\pm$
- Continuity of P $\rightsquigarrow$ Evolution equation for $q_{\rm i}$

Vertical walls

- Continuity of Q $\rightsquigarrow$ Boundary condition for the exterior equations
- ~~Continuity of H~~ $\rightsquigarrow$ $x_\pm = \pm R$ are fixed!
- ~~Continuity of P~~ $\rightsquigarrow$ Evolution equation for $q_{\rm i}$????

$$\begin{cases} \partial_t \zeta + \partial_x Q = 0, \\ \partial_t Q + \partial_x \left(\frac{1}{H} Q^2 + \frac{1}{2} g H^2 \right) = -\frac{1}{\rho} H \partial_x \underline{P} \end{cases}$$

with $\underline{P} = P_{\rm atm}$ in $\mathcal{E}$ and $\underline{P} = P_{\rm i}$ in $\mathcal{I}$.

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- Local conservation of energy

$$\partial_t \mathfrak{e} + \partial_x \mathfrak{F} = 0$$

with

$$\mathfrak{e} = \frac{1}{2}\big(g\zeta^2 + \frac{1}{H} Q^2\big) \quad \text{and} \quad \mathfrak{F} = Q\big(\zeta + \frac{1}{2}\frac{Q^2}{h^2}\big)$$

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- Total Energy

$$E_{\rm tot} = \int_{\mathcal{E}} \mathfrak{e} + \frac{1}{2}\int_{\mathcal{I}} \big(g\zeta_{\rm w}^2 + \frac{q_{\rm i}^2}{h_{\rm w}}\big)$$

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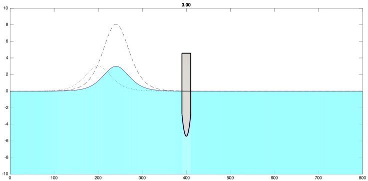
- Conservation of total energy

$$0 = [\![\mathfrak{F}]\!] + \alpha q_{\rm i} \dot{q}_{\rm i}$$

- Evolution equation for $q_{\rm i}$

$$\alpha \dot{q}_{\rm i} = -[\![\zeta + \frac{1}{2}\frac{q_{\rm i}^2}{h_{\rm w}^2}]\!].$$

Including dispersive effects using a Boussinesq model



Fixed object with vertical sidewalls at $x_{\pm} = \pm R$

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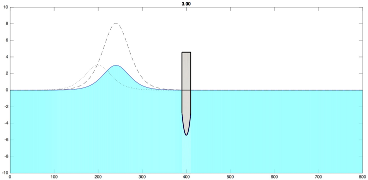

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with

$$\begin{cases} \underline{P} = P_{\mathrm{atm}} & \text{on } \mathcal{E} = (-\infty, -R) \cup (R, \infty), \\ \zeta = \zeta_{\mathrm{w}}(x) & \text{on } \mathcal{I} = (-R, R), \end{cases}$$

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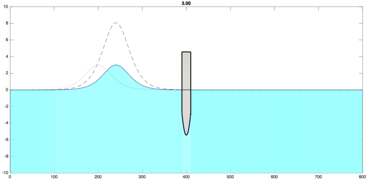

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and one coupling condition

$$q(t, \pm R) = q_{\rm i}(t).$$

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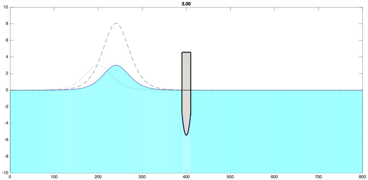

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and one coupling condition

$$q(t, \pm R) = q_{\rm i}(t).$$

and, with an energy conservation argument,

$$-\alpha \dot{q}_{\rm i} = [\![\zeta + \varepsilon \frac{1}{2} \zeta^2 - \delta^2 \partial_x \partial_t q]\!]$$

A transmission problem

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \delta^2 \partial_x^2) \partial_t q + \underbrace{\varepsilon \partial_x (\frac{1}{h_0} q^2) + h \partial_x \zeta}_{:=\Gamma} = 0 \end{cases} \qquad \text{if } |x| > R$$

with transmission conditions

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1. $[\![q]\!] = 0$

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with transmission conditions

1. $[\![q]\!] = 0$
2. $-\delta^2 \partial_t [\![\partial_x q]\!] + [\![\zeta + \varepsilon \frac{1}{2}\zeta^2]\!] = -\alpha \dot{q}_{\rm i}$ with $q_{\rm i}(t) = q(t, \pm R)$.

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- Denote R_0 the inverse of $(1 - \delta^2 \partial_x^2)$ with Dirichlet BC at $x = \pm R$,

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- Denote R_0 the inverse of $(1 - \delta^2 \partial_x^2)$ with Dirichlet BC at $x = \pm R$,

$$\partial_t q = -R_0 \Gamma + \dot{q}_{\rm i} \exp(-\frac{1}{\delta} |x|_R)$$

A transmission problem

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \delta^2 \partial_x^2) \partial_t q + \underbrace{\varepsilon \partial_x (\frac{1}{h_0} q^2) + h \partial_x \zeta}_{:=\Gamma} = 0 \end{cases} \qquad \text{if } |x| > R$$

with transmission conditions

1. $[\![q]\!] = 0$
2. $-\delta^2 \partial_t [\![\partial_x q]\!] + [\![\zeta + \varepsilon \frac{1}{2} \zeta^2]\!] = -\alpha \dot{q}_{\rm i}$ with $q_{\rm i}(t) = q(t, \pm R)$.

- Denote R_0 the inverse of $(1 - \delta^2 \partial_x^2)$ with Dirichlet BC at $x = \pm R$,

$$\partial_t q = -R_0 \Gamma + \dot{q}_{\rm i} \exp(-\frac{1}{\delta} |x|_R)$$

- $\partial_t \partial_x q(\pm R) = -\partial_x R_0 \Gamma \mp \frac{1}{\delta} \dot{q}^{\rm i}$

A transmission problem

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \delta^2 \partial_x^2)\partial_t q + \underbrace{\varepsilon \partial_x (\frac{1}{h_0} q^2) + h \partial_x \zeta}_{:=\Gamma} = 0 \end{cases} \qquad \text{if } |x| > R$$

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- $\partial_t [\![\partial_x q]\!] = -[\![\partial_x R_0 \Gamma]\!] - \frac{2}{\delta} \dot{q}^{\rm i}$

A transmission problem

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ (1 - \delta^2 \partial_x^2)\partial_t q + \underbrace{\varepsilon \partial_x (\frac{1}{h_0} q^2) + h \partial_x \zeta}_{:=\Gamma} = 0 \end{cases} \qquad \text{if } |x| > R$$

with transmission conditions

1. $[\![q]\!] = 0$
2. $-\delta^2 \partial_t [\![\partial_x q]\!] + [\![\zeta + \varepsilon \frac{1}{2} \zeta^2]\!] = -\alpha \dot{q}_{\rm i}$ with $q_{\rm i}(t) = q(t, \pm R)$.

- Denote R_0 the inverse of $(1 - \delta^2 \partial_x^2)$ with Dirichlet BC at $x = \pm R$,

$$\partial_t q = -R_0 \Gamma + \dot{q}_{\rm i} \exp(-\frac{1}{\delta} |x|_R)$$

- $\partial_t \partial_x q(\pm R) = -\partial_x R_0 \Gamma \mp \frac{1}{\delta} \dot{q}^{\rm i}$
- $\partial_t [\![\partial_x q]\!] = -[\![\partial_x R_0 \Gamma]\!] - \frac{2}{\delta} \dot{q}^{\rm i}$
- Use second transmission condition

$$\dot{q}_{\rm i} = \frac{1}{\alpha + 2\delta} \left(\delta^2 [\![-\partial_x R_0 \Gamma]\!] - [\![\zeta + \varepsilon \frac{1}{2} \zeta^2]\!] \right)$$

An ODE!

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + R_0 \partial_x \big(\varepsilon \frac{q^2}{h} + \frac{1}{2\varepsilon}(h^2 - 1) \big) = \dot{q}_{\rm i} \exp(-\frac{|x|_R}{\delta}) \end{cases}$$

with

$$\dot{q}_{\rm i} = \frac{1}{\alpha + 2\delta} \big(\delta^2 [\![-\partial_x R_0 \Gamma]\!] - [\![\zeta + \varepsilon \frac{1}{2} \zeta^2]\!] \big)$$

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Remark

Can be put in conservative form

$$R_0 \partial_x = \partial_x R_1$$

where R_1 is the inverse of $(1 - \delta^2 \partial_x^2)$ with Neumann BC.

An ODE!

$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + R_0 \partial_x \big(\varepsilon \frac{q^2}{h} + \frac{1}{2\varepsilon}(h^2 - 1) \big) = \dot{q}_{\rm i} \exp(-\frac{|x|_R}{\delta}) \end{cases}$$

with

$$\dot{q}_{\rm i} = \frac{1}{\alpha + 2\delta} \big(\delta^2 [\![-\partial_x R_0 \Gamma]\!] - [\![\zeta + \varepsilon \frac{1}{2} \zeta^2]\!] \big)$$

Remark

Can be put in conservative form

$$R_0 \partial_x = \partial_x R_1$$

where R_1 is the inverse of $(1 - \delta^2 \partial_x^2)$ with Neumann BC.

Remark

Same approach works for generating boundary conditions: solve the Boussinesq equations on $(0, \infty)$ given an initial data and $\zeta(t, x = 0)$.

References:

General approach:

- D. L. *On the dynamics of floating structures, 2017*

Shallow water:

- 1*D*, non vertical walls: T. Iguchi, D. L., *Hyperbolic free-boundary problems and applications*, 2018
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- 1*D*, relaxation approximation, E. Godlewski, M. Parisot, J. Sainte-Marie, F. Wahl, *Congested shallow water model: roof modelling in free surface flow*, 2018.
- 2*D* radial, vertical walls: E. Bocchi, *Floating objects in shallow water with a radial symmetry*, 2018

Boussinesq:

- Direct numerical coupling, Jiang, *Ships waves in shallow water*, 2001
- Direct numerical coupling, U. Bosi, A.P. Engsig-Karup, C. Eskilsson, M. Ricchiuto, *An efficient unified spectral element Boussinesq model for a point absorber*, 2018
- Dispersive boundary layer approach, D. Bresch, D. L., G. Métivier, *Waves interacting with a partially immersed obstacle in the Boussinesq regime, 2018*
- Application to generating BC: D. L. , L. Weynans, *Generating boundary conditions for a Boussinesq system*, 2018.

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General approach:

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Thanks for your attention